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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following papers: "Transformations Accelerees de l'Education Scientifique Pendant la Revolution Francaise" (Jean Dhombres); "Building on the Knowledge of Students and Teachers" (Thomas P. Carpenter & Elizabeth Fennema); "Hardiesse et Raison des Recherches Francaises en Didactique des Mathematiques" (Colette Laborde); and "Mathematical Literacy for All Experiences and Problems" (Paolo Boero); "Re-Exploring Familiar Concepts with a New Representation" (A. Arcavi & R. Nachmias); "La Construction du Concept de Figure chez les Eleves de 12 Ans" (G. Arsac); "The Role of Conceptual Models in the Activity of Problem Solving" (F. Arzarello); "How to See Equality in the Difference: An Old Problem with Educational Value" (L. Bazzini); "Exploring Children's Perceptions of Multiplication Through Pictorial Representations" (C. Beattys & C. Maher); "Affective Aspects of the Learning of Mathematics" (M. Beharie & Y. Naidoo); "A Conflict and Investigation Teaching Method and an Individualized Learning Scheme: A Comparative Experiment on the Teaching of Fractions" (A. Bell & D. Bassford); "Some Results of a Large Scale Evaluation of the New Syllabus at French College Level" (A. Bodin); "Grade 8 Students' Understanding of Structural Properties in Mathematics" (L.R. Booth); "Unconditional Mathematics" (L. Brandau & K. Richmond); "Comparison of Teacher and Pupil Perceptions of the Learning Environment in Mathematics" (M. Carmeli, D. Ben-Chaim, & B. Fresko); "Numeracy Without Schooling" (T.N. Carraher); "Strategies and Error Patterns in Solving Rotation Transformation" (C. Chien); "A Solo Mapping Procedure" (K.F. Collis & J.M. Watson); "Le Discours Justificatif en Mathematique: L'implication du Locuteur selon la Representation du Referent" (D. Coquin-Viennot); "Qualitative and Quantitative Time Reasoning in Children" (J. Crepault & S. Samartzis); "The Multidimensional Nature of the Pre-Concepts of Number" (C. Dassa, J.C. Bergeron, & N. Herscovics); "Protocols of Actions as a Cognitive Tool for Knowledge Construction" (W. Dorfler); "Representations du Fonctionnement d'une Procedure Recursive en Logo" (C. Dupuis & D. Guin); "Langage et Representation dans l'Apprentissage d'une Demarche Deductive" (R. Duval); "Towards a Theory of Transition" (N.F. Ellerton & M.A. Clements); "Validation d'un Logiciel d'Aide a la Resolution de Problemes Additifs" (M.C. Escarabajal & M. Kastenbaum); "How Big is an Infinite Set? Exploration of Children's Ideas" (R. Falk & S. Ben-Lavy); "Hypothetical Reasoning in the Resolution of Applied Mathematical Problems at the Age of 8-10" (P.L. Ferrari); "Two Different Views of Fractions:

Fractionating and Operating" (O. Figueras); "Psychological Difficulties in Understanding the Principle of Mathematical Induction" (E. Fischbein & I. Engel); "Incipient 'Algebraic' Thinking in Pre-Algebra Students" (A. Friedlander, R. Hershkowitz, & A. Arcavi); and "Development of the Process Conception of Function by Pre-Service Teachers in a Discrete Mathematics Course" (E. Dubinsky, J. Hawks, D.A. Nichols). Includes a listing of working group and discussion group topics and organizers. (MKR)

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ACTES DE LA 13^{e} CONFÉRENCE INTERNATIONALE

PSYCHOLOGY OF MATHEMATICS EDUCATION

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9-13 juillet 1989

P.M.E. 13

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Bicentenaire de la Révolution Française

Volume 1

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ACTES DE LA 13EME CONFERENCE INTERNATIONALE

PSYCHOLOGY OF MATHEMATICS EDUCATION

Paris, France

9-13 Juillet 1989

PME 13

Volume 1

TABLE OF CONTENTS

page

Table of Contents 1
History and Aims of the International Group Psychology of Mathematics Education 9
Sponsors 12
Introduction by Gérard Vergnaud, Michèle Artigue and Janine Rogalski 13
Working Groups 15
Discussion Groups 16

Plenary Addresses

DHOMBRES J. L'atelier obscur - Transformations accelerées de l'education scientifique pendant la Revolution Française 18
CARPENTER T.P., FENNEMA E. Building on the knowledge of students and teachers 34
LABORDE C. Hardiesse et raison des recherches françaises en didactique des mathématiques 46
BOERO P. Mathematical literacy for all experiences and problems 62

Contributions :

Volume 1

ARCAVI A., NACHMIAS R. Familiar concepts with a new representation. 77
ARSAC G. La construction du concept de figure chez les élèves de 12 ans 85
ARZARELLO F. The role of conceptual models in the activity of problem solving 93
BAZZINI L. How to see equality in the difference: an old problem with educational value 101

BEATTYS C., MAHER C. Exploring children's perceptions of multiplication through pictorial representations 109

BEHARIE M., NAIDOO Y. Affective aspects of the learning of mathematics 117

BELL A., BASSFORD D. A conflict and investigation teaching method and an individualised learning scheme - A comparative experiment on the teaching of fractions 125

BODIN A. Some results of a large scale evaluation of the new syllabus at french college level 133

BOOTH L.R. Grade 8 students' understanding of structural properties in mathematics 141

BRANDAU L., RICHMOND K. Unconditional mathematics 149

CARMELI M., BEN-CHAIM D., FRESKO B. Comparison of teacher and pupil perceptions of the learning environment in mathematics 157

CARRAHER T.N. Numeracy without schooling 164

CHIEN C. Strategies and error patterns in solving rotation transformation 172

COLLIS K.F., WATSON J.M. A solo mapping procedure 180

COQUIN-VIENNOT D. Le discours justificatif en mathématique: l'implication du locuteur selon la représentation du référent 188

CREPAULT J., SAMARTZIS S. Qualitative and quantitative time reasoning in children 196

DASSA C., BERGERON J.C., HERSCOVICS N. The multidimensional nature of the pre-concepts of number 204

DORFLER W. Protocols of actions as a cognitive tool for knowledge construction 212

BEST COPY AVAILABLE

DUPUIS C., GUIN D. Représentations du fonctionnement d'une procédure récursive en Logo 220

DUVAL R. Langage et représentation dans l'apprentissage d'une démarche déductive 228

ELLERTON N.F., CLEMENTS M.A. (Ken) Towards a theory of transition 236

ESCARABAJAL M.C., KASTENBAUM M. Validation d'un logiciel d'aide à la résolution de problèmes additifs 244

FALK R., BEN-LAVY S. How big is an infinite set? Exploration of children's ideas 252

FERRARI P.L. Hypothetical reasoning in the resolution of applied mathematical problems at the age of 8-10 260

FIGUERAS O. Two different views of fractions: fractionating and operating 268

FISCHBEIN E., ENGEL I. Psychological difficulties in understanding the principle of mathematical induction 276

FRIEDLANDER A., HERSHKOWITZ R., ARCAVI A. Incipient "algebraic" thinking in pre-algebra students 283

Volume 2

GALLOU-DUMIEL E. Logo et symétrie centrale 3

GIMENEZ J. About continuous operator subconstruct in rational numbers 10

GOLDIN G.A. Constructivist epistemology and discovery learning in mathematics 15

GRASSI M.G. Comparative analysis of two arithmetic situations 23

GURTNER J.L. Understanding and discussing linear functions in situations. A developmental study 31

HADASS R. Low mathematics achievers' test anxiety 39

HANNA G. Proofs that prove and proofs that explain 45

HAREL G., BEHR M., POST , LESH R. Fishbein's Theory : A
further consideration 52

HART K., SINKINSON A. They're useful - Children's view of
concrete materials 60

HASEMANN K. Children's individuality in solving fraction
problems 67

HAUSSMANN K., REISS M. Aspects of declarative knowledge on
control structures 75

HERAUD B. A conceptual analysis of the notion of length and
its measure 83

HERSCOVICS N., BERGERON J.C. The kindergartners' understanding of cardinal number: an international study 91

HILLEL J. Learning about isosceles triangles 99

HITT F. Construction of functions, contradiction and proof .. 107

HOYLES C., NOSS R., SUTHERLAND R. A Logo-based microworld
for ratio and proportion 115

HOZ R., HAREL G. The facilitating role of table forms in
solving algebra speed problems: real or imaginary? 123

JAIME A., GUTIERREZ A. The learning of plane isometries from
the viewpoint of the Van Hiele model 131

JANVIER C. Representation and contextualization 139

JAWORSKI B. To inculcate versus to elicit knowledge 147

JURDAK M. Van Hiele levels and the solo taxonomy 155

KIERAN C. A perspective on algebraic thinkings 163

KRUMHOLTZ N. A structural conceptual model for investigating
some cognitive aspects of problem-solving 172

KUCHEMANN D. The effect of setting and numerical content on
the difficulty of ratio tasks 180

BEST COPY AVAILABLE

KUYPER H., OTTEN W. Satisfaction and regret about the choice of math 187

KYNIGOS C. Intrinsic versus euclidean geometry: is the distinction important to children learning with the turtle? 194

LACASSE R., GATTUSO L. Mathophobia: A classroom intervention at the college level 202

LARHER A., GRAS R. Le micro-ordinateur, outil de révélation et d'analyse de procedures dans de courtes démonstrations de géometrie 210

LEDER G.C. Gender differences in mathematics learning revisited 218

LEMOYNE G., CONNE F. La résolution de problèmes dans l'enseignement des mathématiques: compte-rendu d'une expérience auprès d'enseignants du primaire 226

LIN F.L. Strategies used by "adders" in proportional reasoning 234

LINCHEVSKY L., VINNER S. Canonical representations of fractions as cognitive obstacles in elementary teachers 242

MANSFIELD H., HAPPS J. Using concept maps to explore students' undertanding in geometry 250

MARIOTTI M.A. Mental images: some problems related to the development of solids 258

MARTIN W.G., HAREL G. The role of the figure in students' concepts of geometric proof 266

MASON J.H., DAVIS P.J. The inner teacher, the didactic tension, and shifts of attention 274

MAURY S., JANVIER M., BAILLE J. Lecture et construction de diagrammes en bâtons dans le premier cycle de l'enseignement secondaire français 282

MCRAE L., HARRISON B. Cooperative group learning of geometric proof construction: a classroom assessment 290

MCLEOD D., METZGER W., CRAVIOTTO C. Comparing experts' and novices' affective reactions to mathematical problem solving: an exploratory study 296

MITCHELMORE M.C. The development of children's concepts of angle 304

Volume 3

MURRAY H., OLIVIER A. A model of understanding two-digit numeration and computation 3

NADOT S. The computer produces a special graphic situation of learning the change of coordinate system 11

NANTAIS N., HERSCOVICS N. Epistemological analysis of early multiplication 18

NASSER L. Are the Van Hiele levels applicable to transformation geometry ? 25

NESHER P., SUKENIK M. Intuitive and formal learning of ratio concepts 33

NEUMAN D. Early conceptions of subtraction 41

NOHDA N., ISHIDA J., SHIMIZU K., YOSHIKAWA, REYS R.E., REYS B.J. Computational, estimation performance and strategies used by select fifth and eighth grade japanese students........ 49

OLIVE J. Associations among high school students' interactions with Logo and mathematical thinking 57

OSTA I. Graphic constructions with computer to learn 3D reference system 65

PACE J.P. Applied problem solving in intuitive geometry 74

BEST COPY AVAILABLE

PALLASCIO R., TALBOT L., ALLAIRE R., MONGEAU P. L'incidence de l'environnement sur la perception et la représentation d'objets géométriques ... 82

PARMENTIER C. Angles et pixels - Quelle synergie à 9 ans ? ... 90

PATRONIS T. Interaction by open discussion and "scientific debate" in a class of 12-years old pupils ... 98

PELED I., MUKHOPADHYAY S., RESNICK L.B. Formal and informal sources of mental models for negative numbers ... 106

PESCI A. Inverse procedures: The influence of a didactic proposal on pupils' strategies ... 111

PIRIE S., KIEREN T. Through the recursive eye: mathematical understanding as a dynamic phenomenon ... 119

PRESMEG N., FRANK A. Cognitive aspects of the learning of mathematics in a multicultural school ... 127

REISS M. Qualitative and quantitative predictions as determinants of system control ... 135

SCHWARZ B., DREYFUS T. Transfer between function representations: a computational model ... 143

SFARD A. Transition from operational to structural conception: the notion of function revisited ... 151

SHUMWAY R. Supercalculators and research on learning ... 159

SIERPINSKA A., VIWEGIER M. How and when attitudes towards mathematics and infinity become constituted into obstacles in students ? ... 166

SMITH J., ARCAVI A., SCHOENFELD A.H. Learning Y intercept: Assembling the pieces of an "atomic" concept ... 174

SNIR N., MEVARECH Z., MOVSHOVITZ-HADAR N. Computers, video, both or neither: which is better for teaching geometry ? ... 182

STRASSER R., BROMME R. Vocational mathematics teachers' cognition of mathematical and vocational knowledge 189

STONEWATER J.K. Training elementary teachers in problem solving strategies: impact on their students' performance 197

SUTHERLAND R. Developing algebraic understanding: the potential of a computer based environment 205

THOMAS M., TALL D. Verbal evidence for versatile understanding of variables in a computer environment 213

TIROSH D., ALMOG N. Conceptual adjustments in progressing from real to complex numbers 221

VERSCHAFFEL L., DE CORTE E., PAUWELS A. Does the semantic structure of word problems affect second graders' eye-movements ? 228

VINNER S. The lesson - A preconceptional stage 236

YACKEL E., COBB P., WOOD T. An analysis of the emotional acts of young children while learning mathematics 244

YERUSHALMY M. The use of graphs as visual interactive feedback while carrying out algebraic transformations 252

ZAZKIS R., LERON U. Images of geometrical transformations: from Euclid to the turtle and back 261

ZEHAVI N., SCHWARZ B. A knowledge-base of student reasoning about characteristics of functions 269

List of addresses 277

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HISTORY AND AIMS OF THE INTERNATIONAL GROUP

PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

P.M.E. came into existence at the Third International Congress on Mathematical Education (I.C.M.E.3) held in Karlsruhe, Germany, in 1976. Its past presidents have been Pr Efraim Fischbein of Tel Aviv University, Pr Richard R. Skemp of Warwick University, Dr Gérard Vergnaud of the Centre National de la Recherche Scientifique in Paris, Pr Kevin F. Collis of the University of Tasmania and Pr Pearla Nesher of the University of Haifa.

The major goals of the Group are:

- to promote international contacts and the exchange of scientific information in the psychology of mathematics education;

- to promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;

- to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Present officers of the group

Président	Nicolas Balacheff (France)
Vice-President	Willibald Dörfler (Austria)
Secretary	David Pimm (U.K.)
Treasurer	Carolyn Kieran (Canada)

Other members of the International Committee are:

Alan Bishop (U.K.)
George Booker (Australia)
Terezina Carraher (Brasil)
Paul Cobb (U.S.A.)
Theodore Eisenberg (Israel)
Fou Lai Lin (Taiwan)
Christine Keitel (F.R.G.)
Colette Laborde (France)
Franck Lester (U.S.A.)
Teresa N. di Mendicutti (Mexico)
Nobuhako Nohda (Japan)
Joao Ponte (Portugal)
Janos Suranyi (Hungary)

PME holds an international conference every year.
The Program Committee for PME 13 was the following:

Michele Artigue (France)
Alan Bishop (U.K.)
George Booker (Australia)
Christine Keitel (F.R.G.)
Colette Laborde (France)
Teresa N. di Mendicutti (Mexico)
Janine Rogalski (France)
Gerard Vergnaud (France)

Conference convener : Gerard Vergnaud
Conference secretary: Catherine Lavigne

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Last year, the International Committee discussed the main ideas that should preside at the selection of papers and give the scientific orientation of PME. They read as follows.

To achieve the goals of PME, the focus of our future activities should be related to scholarly inquiry and the development and analysis of theoretical perspectives on the teaching and learning of mathematics.

1. By specifying the teaching and learning of mathematics we are primarily interested in the interactions among students, teachers and the learning environment. These interactions are governed by the manner in which students' work is specified or contracted. The social conditions of the teaching/learning environment affect how mathematical meaning is negotiated between teachers and students. These constraints inevitably have an impact on our research.

2. Special attention should be given to the nature of mathematical knowledge. Epistemological analysis of mathematical concepts aims to clarify their meaning through the study of their historical development and the obtacles encountered in their construction and acceptance. Epistemological analysis of the teaching situation aims to identify characteristics in the learning of particular mathematical concepts and the obstacles encountered by pupils in their reconstruction.

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3. The focus on the teaching and learning of mathematics necessitates an interdisciplinary approach. This has led to the emergence of a community of scholars which is more than just a grouping of psychologists, mathematicians and mathematics educators. The discipline of the Psychology of Mathematics Education has its own conceptual frameworks, theoretical approaches, and its particular methodological emphasis.

4. We recognise that there are special conceptual and methodological issues in doing research in classroom settings. Conceptual models and methods of inquiry simply borrowed from other disciplines may be inadequate for classroom research. Therefore, investigations within the Psychology of Mathematics Education should develop new paradigms and formulate new interdisciplinary perspectives on research. For this research to have impact on practice, appropriate means of involving practitioners and communicating implications are essential.

The meetings of the International Group for the Psychology of Mathematics Education promote the critical analysis and debate of inquiry. Different views about the teaching and learning of mathematics are valued and different forms of communication are encouraged. However, the relationships between research, theory and practice should be the major focus of PME presentations.

SPONSORS

Centre National de la Recherche Scientifique

Ministère de la Recherche et de la Technologie

Ministère de l'Education Nationale

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INTRODUCTION

by Gerard Vergnaud, Janine Rogalski and Michèle Artigue

When we offered the possibility to organize PME 13 in Paris in 1989, on the bicentenary of the French Revolution, we meant to stress the idea that a new challenge is now open to our societies.

The French Revolution took place at the turn of the Industrial Revolution; our societies are now faced with the Scientific Revolution. Not only do we need scientific elites as was the case when Lagrange, Laplace, Monge, Condorcet, Borda, Lavoisier and others obtained the creation of the first "Great Schools": Polytechnique and l'Ecole Normale Superieure, but also is it the case today, that workers, farmers and employees, at all levels, need a good scientific training.

During the French Revolution, there was already a great movement to alphabetize people, and to promote the decimal system all over France (see Dhombres' conference, this volume). Today the scientific, technological, economical and social conditions make it necessary for everybody to understand more advanced mathematical concepts. This requires the improvement of the training of teachers in mathematics; also a better understanding of the relationship of mathematics to the other disciplines, and to the different ways mathematics is used in different professions.

A psychological approach to mathematics education is obviously needed, to trace mathematical ideas in students' behaviors, errors, wordings and difficulties, and to study social interaction in the classroom. This cannot be done efficiently without a sound knowledge of mathematics and its epistemology. It is particularly important to have a clear understanding of the relationship between the problems and situations tackled and the emergence of new mathematical concepts, procedures and reasonings; also to take into account the conditions under which such emergence takes place.

PME is an exceptional place for the development of such considerations, as it makes it possible for researchers with different backgrounds and interests, to meet and exchange ideas and results. From year to year we can observe the positive effects of this cooperation. This year, the set of papers offered to PME 13 participants confirms some tendancies that had emerged at the preceding meetings. There are more papers on the learning and teaching of mathematics at the University level and the late secondary level; also on the social dimension of mathematics education; and more and more papers on the use and impact of a computer environment.

A special attention is now paid to the teacher as an actor of the educational process. As it is a new topic for research, there are only a few research reports; but two discussion groups are devoted to it. They should favor the emergence of a theoretical and methodological framework to approach this problem as a research question and therefore bring new lights to the important problem of teacher training.

WORKING GROUPS

A Social Psychology of Mathematics Education

A. Bishop, N. Hadar, C. Keitel, C. Laborde

B Geometry

H. Mansfield, G. Becker, A. Bell, A. Guttierez

C Micromath Research Methodology

N. Zehavi

D Advanced Mathematical Thinking

D. Tall, E. Dubinsky, T. Dreyfus, J. Mason, G. Harel,
G. Ervynck, J. Duffin

E The Role of Representations in the Acquisition of
Mathematical Knowledge

F. Lowenthal

F Ratio and Proportion

K. Hart

The address of the first coordinator can be found in the list of addresses, at the end of volume 3.

DISCUSSION GROUPS

1. Teacher as researchers in Mathematics Education.

   S. Lerman

2. Algebraic Thinking.

   C. Kieran, A. Arcavi, A. Friedlander, R. Sutherland, D. Wheeler.

3. The nature of inconsistent thought and implications for teaching.

   E. Fischbein, G. Harel, D. Tirosh, A. Graeber, D. Tall, P. Wilson

4. Tacit models in mathematical reasoning: educational implications.

   E. Fischbein, S. Vinner, D. Tall, P. Wilson.

5. Classroom Research.

   J. Van den Brink, N. Ellerton, K. Crawford, L. Joffe, D. Siemon, L. Streefland.

6. Mathematics Teacher Education Research.

   B. Underhill, C. Maher, P. Ernest, M. Simon, N. Ellerton.

7. Evaluating theories of mathematical understanding.

   S. Pirie, T. Kieren, N. Herscowicz, J. Bergeron

8. What is Research ?

   K. Hart, T. Carraher, O. Figueras, F. Lester

9. Computer microworld and multiple mathematical sign system.

   C. Hoyles, E. Filloy, R. Noss, J. Rojano, R. Sutherland

10. Psychology of Inservice Education of Maths Teachers. A research perspective.

    J. Ainley, M. Askew, S. Dawson, B. Jaworski

11. Learning mathematics and cultural context.

B. Denys, M. Dupraz, H. Emori, B. Parzysz, M. Yamauchi

12. Theoretical and practical aspects of proof.

G. Hanna, N. Balacheff, D. Chazan, D. Alibert.

The address of the first coordinator can be found in the list of addresses, at the end of volume 3

L'ATELIER OBSCUR
Transformations accélérées de l'éducation scientifique pendant la Révolution française

Jean Dhombres

Ecole des hautes études en sciences sociales, Paris

Au sortir de la Révolution , et bien conscient de la nouveauté du monde qu'elle avait permis d'instaurer, un savant français s'efforçait à saisir la place que les sciences avaient pu jouer dans la décennie pendant laquelle fut institué le citoyen et renversé la monarchie . Chez un mathématicien, ce recours à l'histoire signait une autre nouveauté, et si l'exercice se présentait comme une dramaturgie, c'est que celle-ci s'avérait indispensable pour libérer les esprits. Il fallait non seulement forger le jugement qu'une communauté portait a posteriori sur des événements proches -mais éminemment conflictuels -à l'occasion desquels elle avait joué un rôle reconnu, mais aussi fixer pour l'avenir la nature des relations que cette communauté entretiendrait désormais avec le pouvoir, sinon avec la société. Récit qui ne peut nous laisser indifférent, d'autant que ce fut la voix presque innocente d'un jeune homme qui se fit entendre. En élaborant de la sorte son histoire, la communauté savante n'avait-elle pas les yeux tournés vers l'avenir ? Récemment élu à la première classe de l'Institut où il succédait à l'astronome Delambre (promu secrétaire perpétuel), et après avoir brillé dans la première promotion de l'Ecole polytechnique (réunie dans les premiers mois de l'année 1795), Jean Baptiste Biot s'avérait le chantre idéal.

Pour décrire ce passé révolutionnaire et pour vanter le triomphe des Lumières dans les manifestations de la science de l'an 2 et de l'an 3 , Biot inventait une étonnante métaphore , celle de " *l'atelier obscur* ". L'image(1) , aussi inattendue que belle, avait l'avantage de faire apparaître les savants comme des artisans appliqués au travail, collectivement responsables des réussites techniques mises au service de la défense nationale, mais individuellement trop obscurs pour qu'on les accusât de toute participation a des factions politiques ou, *horresco referens* , à l'organisation de la Terreur. L'obscurité de la forge cache et protège le forgeron, mais elle exalte le feu qui dompte le fer et crée l'outil. Quel était ce feu auprès duquel les savants s'activaient ? Quels étaient les outils qu'ils préparaient? Comment formaient-ils leurs tâcherons ?

Les réponses à ces questions ne sont pas indépendantes et nous révèlent la pédagogie d'une époque, tant il est vrai que les savants du temps explicitèrent eux-mêmes les liaisons. Une pratique pédagogique nouvelle était en place. Pour mieux faire entrer dans les esprits ce que le compte-rendu de Biot avait déjà martelé en 1803, un autre mathématicien écrivait deux ans plus tard un Essai sur l'enseignement en général et sur celui des mathématiques en particulier. La vraie spécialité de Sylvestre François Lacroix était l'enseignement , et son rôle était d'autant mieux assuré qu'il disposait de la reconnaissance de ses pairs en siégeant lui aussi à la première classe de l'Institut (celle attribuée aux scientifiques).

On comprendra encore mieux l'importance dévolue à l'éducation si l'on ajoute que la Révolution dota la majorité des savants d'un métier de professeur. A la carrière d'un d'Alembert , disparu en 1783 et qui n'enseigna jamais à titre professionnel, s'oppose par exemple celle du naturaliste Georges Cuvier, professeur au Muséum d'Histoire Naturelle depuis son arrivée à Paris en 1795, ou encore celle du mathématicien et mécanicien Siméon Denis Poisson, qui exerça l'enseignement dès sa sortie de l'Ecole polytechnique. Mais nous n'allons pas visiter l' "*atelier obscur* " en rédigeant des notices biographiques sur des enseignants, ou en nous félicitant d'ancêtres aussi ardents que glorieux. Certes le Bicentenaire de la Révolution Française nous y inciterait - ne sommes nous pas très proches du 14 juillet ?- mais si nous reprenons les faits et les gestes d'un certain nombre des artisans de la science, c'est surtout parce que dans cet "*atelier obscur* " se forgèrent des traits constitutifs de l'enseignement scientifique moderne . D'autres choses furent également préparées, mais bientôt effacées par le temps. De cette confrontation doit naitre la réflexion. Entrons donc dans notre passé.

L'atelier des lois: le mètre du maître

Le 10 décembre 1799 (19 frimaire an 8), la loi sanctionnait les définitions du mètre-étalon et du kilogramme-étalon : l'uniformité des poids et mesures manifestée par le système métrique devenait partie intégrante de la Constitution française. Le ministre de l'Intérieur qui avait hâté cette adoption n'était pas un politique ordinaire, mais le célèbre mathématicien Laplace que Bonaparte venait de s'associer au lendemain du coup d'état de Brumaire. Il voulait sans doute bénéficier de l'aura qui entourait l'auteur de la Mécanique céleste dont les deux premiers tomes venaient juste d'être publiés. On parlait, dans les journaux, du "*Newton français* ". En tout cas, la présence de Laplace symbolise de façon remarquable l'action directe, sinon pressante, des savants qui n'hésitaient pas à bouleverser la vie ordinaire de leurs concitoyens

Outre son uniformité, que des hommes politiques pouvaient aisément défendre, le système promulgué présentait une rigueur scientifique contraignante, celle de la décimalisation qui ordonnait les multiples et les sous-multiples de toutes les unités selon la litanie que nous avons apprise sur les bancs de l'école primaire: litre, décilitre, centilitre, millilitre et de l'autre côté, décalitre, hectolitre ; tout comme s'ordonnait la série des longueurs: millimètre, centimètre,décimètre, etc. Cette rigueur portait la marque indéniable de " *l'atelier* " des savants. Il n'y avait pourtant là rien d'obscur, car ceux-ci imposaient désormais à tous ce qui avait la clarté de la loi . Mais il n'est pas anodin de remarquer qu'à lui seul le principe de la décimalisation conduisait inéluctablement à changer les noms en usage. Car si l'on pouvait à la rigueur concevoir d' uniformiser les mesures en conservant par exemple le pied dans le vocabulaire des longueurs, quitte à fournir une définition nouvelle et nationale de sa valeur effective, il aurait été particulièrement gênant de prendre le pouce comme sa dixième partie alors que toute la tradition voulait qu'il en fut la douzième. La décimalisation imposait l'introduction de noms nouveaux : par exemple le mètre (un néologisme tiré du grec signifiant la mesure) ou le gramme, avec lesquels nous sommes tellement familiarisés aujourd'hui qu'il nous faut faire effort pour en rappeler l'étrangeté. Celle-ci éclate cependant si l'on rappelle le cade qui fut un temps utilisé vers 1793. Bref, tout un vocabulaire nouveau entrait dans la vie du citoyen ordinaire, et sa sémantique savante était conçue pour éradiquer les conjugaisons des rapports si divers entre les mille, stade, perche, tonneau, setier, boisseau, maille, grain et autre pinte.

Bien des savants tenaient essentiellement à la décimalisation et, dès avant 1789, Lavoisier avait indiqué l'avantage de son adoption pour mieux souder la communauté scientifique mondiale. Mais il ne cherchait pas à imposer les unités de repérage: " *Les chimistes de toutes les parties du monde pourraient sans inconvénients se servir de la livre de leur pays, quelle qu'elle fut, pourvu que au lieu de la diviser, comme on l'a fait jusqu'ici, en fractions arbitraires, on se déterminât par une convention générale, à la diviser en dixièmes, en centièmes, en millièmes* "(2). Aussi ce fut sans difficulté que l'Académie des sciences rendit rapport le 19 mars 1791 et proposa le système décimal " *qui répond à l'échelle arithmétique* " et du coup doit "*être préféré pour les mesures d'usage* ". Cette décision fut adoptée avec célérité par la Législative, le 30 mars du même mois, en même temps que des propositions étaient faites pour la valeur des unités fondamentales elles-mêmes. L'unité de longueur serait la dix millionième partie du quart du méridien terrestre dont la détermination suffisamment précise était aussitôt lancée par une équipe composite de savants supervisés par l'Académie des sciences. Un minimum de concertation paraissait toutefois indispensable pour qu'une telle décision trouve le consensus des pays voisins, et l'on pouvait espérer qu'en prenant le temps, et en y mettant quelque doigté, les liens récemment forgés par les astronomes français et anglais à l'occasion de la jonction géodésique entre l'Observatoire de Greenwich et celui de Paris emporteraient l'assentiment.

Aussi surprenant que cela puisse paraître, ce ne furent pas les politiques qui poussèrent à l'adoption immédiate d'un nouveau système, mais les savants eux-mêmes, au risque de braquer l'opinion des académies étrangères. A leurs yeux, la qualité des mesures géodésiques réalisées au cours du XVIIIe siècle permettait sans difficulté de fixer un mètre "provisoire". Or puisque le plus difficile était de rendre transparentes les nouvelles échelles décimales, il était préférable d'agir au plus vite, pour profiter du vent de réforme qui balayait la France entière. Les savants acceptèrent du coup de se prêter au jeu de la propagande, car au-delà même d'un système national et normatif, ils s'engageaient à diffuser cet esprit rationnel fait de précision qui était intellectuellement profitable à tous. Ils rédigèrent donc des rapports explicatifs que le gouvernement s'empressa de diffuser. Le fondateur de la classification des cristaux selon une base géométrique rationnelle, René Just Haüy , pourtant emprisonné quelque temps à l'été 1792, composa une *Instruction sur les mesures déduites de la grandeur de la terre, uniformes pour toute la République, et sur les calculs relatifs à leur division décimale*, sortie avec le printemps , le 1er avril 1794. Malheureusement, l'excellent physicien ne savait pas être court en restant exact, de sorte que son instruction culminait à 224 pages, ce qui était beaucoup trop si l'on envisageait une diffusion vraiment populaire. Malgré les récriminations dans les départements, l'*Instruction abrégée* réalisée dix jours plus tard comportait encore 147 pages.

Ainsi, sortant de leurs laboratoires et de leurs académies, les savants se heurtaient de front à la difficulté de la popularisation. Mais ils n'avaient guère la possibilité de tergiverser, car les responsables politiques leur enjoignaient sans ambages de faire court , simple, et clair. En cherchant à être utiles, tout en rendant visible l'inscription de la science dans la Nation, la communauté scientifique adoptait sans rechigner le rythme particulièrement rapide de la vie politique révolutionnaire, notamment jacobine. On ne compta aucun abandon, et les savants s'adaptèrent à la situation. Certains cherchèrent, notamment par des méthodes graphiques et des tableaux, à rendre plus visibles les relations précises de toutes les mesures anciennes avec les nouvelles . Il serait intéressant pour un pédagogue de retracer ces essais didactiques dans leur variété entre 1795 et 1800, et on pourrait peut-être en déduire les voies d'apprentissage préférées à cette époque. On trouverait d'ailleurs dans ces textes le moule de bien des exercices de calcul qui figureront dans la besace des instituteurs de la IIIè République. Il y eut en tout cas une floraison d'ouvrages , puisqu'en 1799 environ 13% des titres nouvellement imprimés en France et portant sur des matières scientifiques étaient spécifiquement consacrés à la métrologie. Mais dès 1803 ce pourcentage tombait de moitié, non que les nouvelles mesures aient été remises en cause, mais parce que les manuels scientifiques les incorporaient désormais dans l'arsenal éducatif (3). Les manuels anciens furent remaniés. A titre d'exemple, prenons celui de F.P.Silvestre, qui était d'abord sorti en 1787 puis édité à nouveau en 1809 " *à l'usage des pensionnats et des écoles chrétiennes* ". Si l'auteur ne se résolvait pas à abandonner la livre d'un poids de 16 onces, le marc de 8 onces, l'once de 8 gros, le gros de 3 deniers ou scrupules et le scrupule de 24 grains ou deux oboles etc, il n'en indiquait pas moins avec netteté que " *les savants ont proposé et le gouvernement a adopté* " des mesures nouvelles dont il dressait la liste et dont il fournissait le schéma explicite. Cependant il fallait aller au-delà des lecteurs de livres scientifiques comme des élèves des écoles secondaires. Particulièrement significatif est entre 1790 et 1800 le doublement des titres de manuels arithmétiques à l'usage des marchands. (4) Ce système métrique était effectivement défendu bec et ongles par le monde savant qui s'avérait militant et productif.

La nouveauté même de ce système rendait difficile son emploi immédiat et généralisé ; elle suggérait une autre méthode. Elle rendait possible l'action auprès des plus jeunes dont on devait attendre un effet de contagion. Aussi sa diffusion fut canalisée par l'enseignement, avec forte pression auprès des maitres afin qu'en utilisant le système métrique à l'exclusion de tout autre, ils le propagent. Ecoutons l'exhortation de Laplace destinée à plus d'un millier d'élèves de l'Ecole normale, ceux qui devaient devenir les instituteurs de la République: " *L'un des plus utiles objets qui vous occuperont, après être retournés dans vos* *départements, sera de faire connaître à vos concitoyens, et spécialement aux instituteurs des écoles primaires, ce bienfait des sciences et de la révolution* "(5). Le savant devenu professeur indiquait ici les deux sources de la modernité, et manifestait de la sorte un véritable militantisme . La rationalité du système métrique paraissait le garant de son universalité , que l'on devait au besoin soutenir par la coercition. On se trouve bien en présence d'une pédagogie engagée, et sa réussite dans le cadre de la France du Directoire donne à réfléchir. Pour les adultes , qui ne pouvaient aller sur les bancs des écoles, le gouvernement prévoyait une forme de pédagogie incitative, qu'explicitait, le 17 pluviôse an 9, le chimiste Chaptal devenu à son tour ministre de l'Intérieur du premier Consul. Il commentait à l'intention des préfets un arrêté sur l'établissement d'une nouvelle administration locale des bureaux de pesage, mesurage et jeaugeage publics, dont les revenus éventuels devaient aller aux communes. La morale et la bienfaisance renforçaient la loi qui devait s'imposer: " *Le gouvernement compte trouver dans l'exécution de son arrêté*

1° Une garantie contre la fraude, que l'infidélité des peseurs et mesureurs fait souvent éprouver au commerce .

2°Une ressource offerte aux communes pour acquitter leurs charges et soutenir les hospices .

3°Un moyen prompt et facile de familiariser les citoyens avec les nouveaux poids et mesures . "

L'atelier modèle: la science ploie l'éducation

En 1798 un professeur de mathématiques à l'Ecole centrale de la rue Antoine employait le mode du présent pour résumer sa confiance dans l'établissement du système métrique: " *La raison et l'utilité commune triomphent à la longue de l'ignorance et de la routine*". (6) Cette lutte contre la routine prenait un caractère positif en soi, et l'homme qui parlait ainsi n'abusait pas de la rhétorique. Tous les savants de l'époque révolutionnaire le disent et le répètent à l'envi : la science était jeune, elle impliquait le changement et c'est d'abord en elle que s'était manifestée une révolution, celle que Lavoisier et son groupe imposèrent à la chimie. La métaphore qui passait sur toutes les lèvres ne s'était pas encore affadie.

Cette lumière ne pouvait rester sous le boisseau, et ses effets devaient être partagés par tous. Il fallait donc faire passer la modernité dans l'enseignement. En préface à un volumineux Traité de calcul différentiel et de calcul intégral de 1797, Lacroix s'était déclaré " *frappé des difficultés que présentait l'étude de l'Analyse et de la Géométrie transcendante, par l'intervalle qui séparait les ouvrages élémentaires les plus étendus, des Mémoires où se trouvaient consignés les nouvelles découvertes* " . Son ouvrage avait pour but d'y remédier. N'allons pas imaginer ici la répétition de la querelle des Anciens et des Modernes, n'allons pas croire que la modernité se justifiait par elle même, comme la mode. C'est parce que les grands géomètres du XVIIIè siècle avaient donné " *une perfection* " à la marche de l'analyse, en élaborant des méthodes générales, qu'il convenait de revoir " *la manière de présenter les vérités connues avant eux* " . Une telle révision pouvait paraître attentatoire à la nature intemporelle de la vérité mathématique , et l'on vit des responsables ministériels prendre le soin d'indiquer que la solidité des méthodes importait dans le dévoilement des objets mathématiques: " *Préférez donc dans l'enseignement les méthodes générales; attachez vous à les présenter de la manière la plus simple; et vous verrez en même temps qu'elles sont presque toujours les plus faciles* " (7) Plus faciles, mais surtout plus fertiles, car on recherchait avant tout l'inventivité, la possibilité d'adapter la généralité d'une méthode à des circonstances variées.

Nous sommes quelque peu blasés devant ce langage pourtant neuf à la fin du XVIIIè siècle, pour avoir déjà entendu ressasser cette rhétorique à propos des " mathématiques modernes " dans les années soixante. L'analogie n'est pas fortuite, et un sociologue aurait tout à fait raison de souligner que, dans ce cas comme lors de la Révolution, il importait de changer d'échelle, et par conséquent de pédagogie. Vers 1960, il fallait entrer dans une éducation secondaire de masse. En 1793, face à l'urgence de la guerre, il était indispensable de ruser avec le temps. Il fallait par exemple faire apprendre plus vite et à un plus grand nombre de gens les procédures de fabrication de la poudre à partir du salpêtre, et la préparation de ce dernier à partir des produits récoltables sur le sol français. Mais l'objectif était délibérément plus ambitieux puisqu'il convenait de rendre les personnes ainsi formées capables de se transformer à leur tour en éducateurs afin de répéter, pour l'avantage de bien d'autres encore, l'apprentissage initialement suivi. Le principe, comme dans une réaction en chaîne, était de bénéficier d'un effet multiplicateur. Telle fut la conception qui présida à la mise en place de l'Ecole des armes en février 1794 (pluviôse an 2) . On a souvent souligné le caractère improvisé de cette formation et, par contre-coup, célébré sa réussite sans peut-être suffisamment chercher à déterminer les facteurs qui la garantirent. Car il faut dépasser le recours à l'enthousiasme révolutionnaire qui, malgré le langage d'un Barère, ne saurait toujours permettre de " casser des briques" : *"L'Ancien régime aurait demandé trois ans pour avoir des écoles, pour former des élèves, pour faire des cours de chimie et d'armurerie. Le nouveau régime a tout accéléré. Il demande trois décades..."*(8)

En fait, en collant à la science en train de se faire, on trouvait un modèle nouveau de formation qui fut adopté à la fois pour l'Ecole normale de l'an III, l'Ecole polytechnique, l'Ecole de Mars et l'Ecole de canonnage et de navigation. Et ce malgré des contenus de formation tout à fait différents les uns des autres puisque les objectifs étaient distincts : former des soldats-techniciens dans les deux derniers cas, des ingénieurs dans le deuxième et des instituteurs dans le premier. La modernité permettait une innovation majeure qui se lit dans le nombre des élèves concernés: environ mille pour les cours sur le salpêtre, trois mille cinq cents pour l'Ecole de Mars (été 1794), environ quatre cents à l'Ecole qui prendra bientôt le qualificatif de polytechnique et peut-être mille quatre cents à l'Ecole normale (toutes les deux ayant fonctionné au début de l'année 1795). Des effectifs considérables qui tranchaient de façon notoire sur ce qui existait dans les dernières décennies de l'Ancien régime. Pour briser la dure contrainte temporelle, on augmentait considérablement les masses en jeu et l'analogie s'impose avec la levée en masse de soldats décrétée le 23 août 1793 par Lazare Carnot, le mathématicien du Comité de salut public.

Mais pour réussir ce mouvement, car telle était la logique intellectuelle présidant au système adopté, il fallait faire appel aux esprits les plus novateurs, c'est-à-dire aux inventeurs, qui étaient en train de se muer en professeurs. Leur intelligence devait pouvoir transmettre non seulement l'essentiel des techniques à utiliser sur le champ, mais en outre infuser l'esprit de rationalité qui avait guidé leurs découvertes. En janvier 1795 , le rédacteur de l'introduction aux cours de l'Ecole normale exprimait fortement ce credo lorsqu'il disait que " les *professeurs seraient mal choisis s'il était besoin de les présenter* "[13]. En mathématique, officièrent Lagrange, Laplace et Monge. C'est dire si l'on pensait que chez les innovateurs la théorie et la pratique étaient indissolublement liées.

De la même façon, les mues récentes de la science ne devaient pas être présentées seulement à l'occasion de "cours révolutionnaires". Rajeunir les écrits apparaissait comme une nécessité. A partir de l'an 2, sur la mode de la litanie, on parla de manuels à rédiger qui présenteraient les différents domaines sous leur jour le plus récent. La logique de la méthode révolutionnaire paraissait ici éclatante puisque les élèves de l'Ecole "normale " -la sémantique du nom est nette- devaient simultanément apprendre les façons nouvelles mais aussi bien préparer les nouveaux livres élémentaires dictant la norme de l'enseignement. Si l'on veut bien mesurer l'essor considérable de la production de manuels scientifiques français sur la décennie suivant 1795, et examiner soigneusement les contenus, notamment en mathématiques , physique, ou histoire naturelle, et si l'on tente de les comparer aux ouvrages précédemment en vogue, on ne peut que constater l'influence majeure de l' Ecole normale[14] . Contentons-nous de signaler à titre d'exemple que l'organisation de l'algèbre "moderne" dans l'enseignement- abandon de la théorie des proportions, traitement semblable et littéral des équations polynomiales des quatre premiers degrés, formule du binôme de Newton et analyse algébrique des séries, écriture analytique de la géométrie des droites, des cercles et des plans- fut déterminée par l'originalité des cours de Laplace , de Lagrange et de Monge au premier semestre de 1795 . S'il est fort dommage que cette façon n'ait guère subi ensuite de changements pendant une cinquantaine d'années, cela ne fait que renforcer le poids prépondérant de la première Ecole normale sur tout l'enseignement scientifique français.

En géométrie par exemple, les manuels nouveaux furent suffisamment nombreux pour que l'on puisse assurer que chaque élève de l'enseignement secondaire en possédait un(10). L'affaire n'est pas de minime importance, car du coup le professeur pouvait rompre avec la tradition des cours dictés dont la plupart des rapports sur les collèges de l'Ancien régime disent la nocivité. Un autre mode d'activité, et par suite une autre pédagogie, pouvaient s'établir dans les classes qui, rompant avec la pratique précédente, étaient organisées par matières dans les Ecoles centrales, ces lieux d'enseignement secondaire (et à plus d'un titre supérieur) créés par la Convention avant de se séparer. De cette distribution découlait une autre conséquence, aussi peu banale, l'individualisation des disciplines- mathématiques, physique et chimie, histoire naturelle- et par la même celle des différents professeurs qui y étaient attachés. Ceux-ci avaient pour mission de représenter la jeunesse de la science puisqu'ils devaient couvrir leur domaine jusque dans ses derniers développements. Nous en avons confirmation aussi bien dans les réponses à une enquête de l'an VII qui fut réalisée par le Directoire auprès de tous les professeurs, que dans les cursus scolaires rédigés par ces mêmes professeurs pour être soumis à un jury d'instruction au moment de leur embauche. Même si la population scolarisée dans les Ecoles centrales était en nette diminution par rapport à celle des anciens collèges, l'autonomie acquise par les sciences grâce à des classes spécialisées, voire la prépondérance de leur enseignement conjuguées avec la possibilité de manuels, put entrainer jusque dans les écoles privées un renouveau de la pratique scientifique , du moins pour cette portion privilégiée des jeunes qui accédait aux classes en question.

L'atelier du vrai: l'éducation ne se décrète pas

Une phrase du jeune physicien Arago à son ami le géographe et naturaliste Alexander von Humboldt fait saisir toute l'exigence de précision et de rigueur de la communauté scientifique : "*Humboldt, tu ne sais vraiment pas écrire un livre. Tu écris sans t'arrêter, mais ce qui en sort n'est pas un livre : c'est un portrait auquel il manquerait le cadre.*" A n'en pas douter, ce cadre était au contraire garanti par la démarche analytique, c'est-à-dire dans le domaine mathématique, par le traitement "algébrique" des problèmes, leur réduction à des équations qu'il s'agissait ensuite de résoudre en adoptant telle technique connue. Seules les méthodes analytiques étaient considérées comme suffisamment générales. Voilà pour l'horizon de l'éducation mathématique: on privilégiait les modes d'attaque plutôt que les objets. Toutefois, l'appropriation de cette voie analytique n'avait rien d'automatique : il fallait faciliter les accès en s'appuyant sur ce qui était déjà connu des élèves et assimilé, par exemple les méthodes géométriques quand bien même elles s'éloignaient de l'analytique. En somme, si la visée de l'enseignement était solidement définie, elle ne déterminait pas une éradication des connaissances précédentes. De nombreux exemples dans les écoles centrales et à l'Ecole polytechnique attestent cette souplesse et nous font saisir la différence avec le comportement des protagonistes des mathématiques dites modernes. A la fin du XVIIIème siècle, en France, une attention pédagogique était proprement à l'oeuvre, alors même que le contenu de l'enseignement changeait. C'est en tout cas une leçon qui est utile aujourd'hui et le présent congrès marque combien l'enjeu est perçu.

Un exemple est significatif, celui de la géométrie descriptive. Monge en dévoilait les règles dès 1795, et la proposait comme un langage, économique autant qu'exact. Il escomptait développer chez les ingénieurs aussi bien que chez les artisans, le souci de la précision et le soin de l'exactitude. Mais tout autant, il entendait rendre visibles les opérations de l'analyse en les transposant sous la forme d'une technique de représentation géométrique. Loin d'opposer à sa démarche celle du pur calcul et de la géométrie différentielle manipulant les symboles, Monge "illustrait". Il était pédagogique au plus profond de sa démarche d'inventeur. Laplace, l'analyste par excellence, avait saisi toute la force de conviction contenue dans cette géométrie et se gardait de la dédaigner puisqu'elle permettait l'analyse : *"La synthèse géométrique a d'ailleurs la propriété de ne jamais perdre de vue son objet, et d'éclairer la route entière qui conduit des premiers axiomes à leurs dernières conséquences."* (11) Monge pouvait dépasser un objectif simplement mathématique et déclarait sans état d'âme au sujet de sa géométrie : *"C'est un moyen de rechercher la vérité : elle offre des exemples perpétuels du passage du connu à l'inconnu, et parce qu'elle est toujours appliquée à des objets susceptibles de la plus grande évidence, il est nécessaire de la faire entrer dans le plan d'une éducation nationale."* (12) La qualité de Monge résidait également dans sa façon de prendre appui sur des gestes répétitifs des étudiants, gestes qui conféraient à l'apprentissage mathématique une coloration d'activité manuelle et presque artisanale. Dans des salles de l'Ecole polytechnique, des élèves devaient tracer les lignes manquantes sur une épure préalablement obtenue, afin d'assimiler les tracés de construction, et ils devaient faire circuler les dessins, pour s'imprégner des techniques un peu comme on se familiarise avec une langue étrangère en l'écoutant souvent. Il n'y avait nulle honte à agir de la sorte : c'est encore un point à méditer aujourd'hui. S'il importe de familiariser le plus grand nombre de gens avec l'argumentation mathématique, il faut aussi se donner les moyens de cette familiarisation.

Pour autant, Monge et les membres du Comité d'Instruction publique n'établirent aucun carcan national qui aurait pu imposer un programme strict dans les écoles centrales. Les professeurs étaient libres et ils devaient seulement respecter le cursus qu'ils avaient eux-mêmes soumis au jury d'instruction. Une liberté semblable était d'ailleurs laissée aux élèves qui pouvaient élire les cours de leur choix. Cette liberté profita largement aux mathématiques puisque cette matière fut choisie, dans toute la France, juste derrière le dessin. Nous pourrions être jaloux de cette faveur. En 1800, les inspecteurs généraux n'existaient pas encore, et à vrai dire leur intervention sous la férule du Premier Consul ne brisa pas véritablement la liberté des contenus, malgré l'imposition de manuels "modernes". C'est qu'il fallut bientôt sous l'Empire élargir les choix effectués et on peut juger des tendances qui parcouraient, en mathématiques, la collectivité des enseignants par la concurrence acharnée que se livrèrent trois manuels au moins. Il y avait celui considéré comme "moderne", dû à Lacroix et conçu vers 1797. Un texte plus ancien résistait bien, dû à Bézout, mathématicien mort en 1783 dont on avait malgré tout rapiécé l'écrit. Intervenait à son tour un volume de conception mélangée, algébrisé certes mais rétrograde en bien des points, dû au polytechnicien Bourcharlat.

Si l'éducation scientifique ne se décrétait pas, il faut pourtant voir qu'elle s'organisait par le haut et que le modèle élitiste -celui de l'Ecole polytechnique- prévalait dans son enseignement, et a fortiori dans l'image qu'elle donnait. Fonctionnait à plein rendement l'émulation entre les enseignants et aussi bien entre les étudiants. Ce fut fécond, pendant un temps.

L'atelier d'aujourd'hui ?

Il serait vain, aujourd'hui, de miser sur un phénomène analogue pour favoriser un renouveau de l'enseignement mathématique. Non que l'émulation soit une mauvaise chose, mais parce que l'on ne dispose plus d'un environnement particulièrement propice aux sciences. Les mathématiques en particulier ne sont-elles pas devenues matière à concours, à classement et de moins en moins objet de curiosité intellectuelle et de culture ? Les sciences, pendant la Révolution, étaient largement commentées dans les journaux généralistes, comme la *Décade philosophique*. Elles étaient appréciées et des savants écrivaient d'extraordinaires ouvrages de popularisation. Les sciences tissaient une nouvelle culture à partir de l'humanisme, culture dont les poètes eux-mêmes se faisaient les hérauts. André Chénier n'écrivait-il pas : "*L'avenir* *appartient aux esprits novateurs.* " Novateurs parce qu'ils recherchaient le vrai, au-delà des considérations plus étroites de spécialités. Si les circonstances extérieures sont moins favorables aujourd'hui, n'appartient-il pas alors aux didacticiens d'aider à retrouver l'allant d'un apprentissage ? Ils ne pourront être utiles qu'en se soumettant à l'exactitude de l'observation et en faisant jouer la pluralité des approches qui mêle psychologues et mathématiciens. Faudra-t-il une Révolution pour que leur atelier, aujourd'hui trop obscur, inscrive son action dans la structure éducative instituée ?

Notes

(1)La citation provient de l'Histoire générale des sciences pendant la révolution française de J.B.Biot (Paris, 1803) : " *Tous les moyens de défense sortirent de l'atelier obscur où le génie des sciences s'était retiré* ".

(2) Lavoisier, Oeuvres complètes. Correspondance, Paris, tome 1, 1953, page 253.

(3) J.Dhombres, Mathématisation et communauté scientifique française (1775 1825), Archives Internationales Hist. Sc., 1986, 117 vol 36, pp.249-293.

(4) J.Hoock, Le phénomène Savary et l'innovation en matière commerciale en France aux XVIIè et XVIIIè siècles, Innovations et renouveaux techniques de l'Antiquité à nos jours, Actes du Colloque International de Mulhouse, septembre 1987, J.P Kintz (ed.), pp.113-123.

(5) P.S Laplace, séance du 11 floréal an III, Séances des Ecoles normales recueillies par des sténographes, et revues par les professeurs, Paris , Régnier, an 3, tome 5.

(6) P. Costaz, Recueil des discours du 1er brumaire an VI, prononcés à l'ouverture de l'Ecole centrale de la rue Antoine. Cité dans J.Dhombres, Etre mathématicien français en 1795, in Horizons mathématiques, Irem de Rennes, 1988.

(7) P.S.Laplace, op. cité à la note (5), séance du 11 germinal an 3, vol 4, pp. 49-50. Cette citation fut reprise en 1800 dans une circulaire ministérielle adressée aux professeurs des écoles centrales.

(8) B. Barère, Discours à la Convention du 18 février 1794. Cité dans Nicole et Jean Dhombres, La naissance d'un pouvoir : sciences et savants en France (1793-1824), Payot, Paris, 1989.

(9) Op. cité à la note (5), tome 1, introduction.

(10) Voir l'ouvrage mentionné à la note (8), chapitre VII.

(11) Laplace, Exposition du système du monde, Paris 1796.

(12) Op. cité à la note (5), tome 1, [illegible] leçon de Monge.

BUILDING ON THE KNOWLEDGE OF STUDENTS AND TEACHERS[1]

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> **From the ground up makes good sense for building.**
> **Beware of from the top down.**
>
> Frank Lloyd Wright

This paper summarizes the findings of a series of integrated studies investigating how teachers use specific research and knowledge about children's thinking to plan and implement instruction in first grade mathematics. Teachers' knowledge and beliefs about children's thinking were found to be related to student achievement, and training focused on understanding children's thinking resulted in significant changes in teachers' beliefs and knowledge about their students, their classroom instruction, and in students' problem solving achievement.

For the last four years we have been studying how teachers use research based knowledge about children's thinking and problem solving to make decisions as they plan and implement instruction, and how this instruction affects their students' learning. Although it may seem obvious that teachers should attend to what their students know in planning and teaching, Clark and Peterson (1986) report that teachers do not tend to base instructional decisions on their assessment of children's knowledge. Our work challenges this premise.

The project was composed of a series of integrated studies. Two baseline studies conducted during the first year of the project examined teachers' knowledge and beliefs about their students' thinking and problem solving and how this knowledge was related to their students' achievement (Carpenter, Fennema, Peterson, & Carey, 1988; Peterson, Fennema, Carpenter, & Loef, 1989). In the second year we conducted an experimental study involving 40 first grade teachers in which we investigated how providing teachers explicit research based knowledge about children's thinking affected their instruction and their students' achievement (Carpenter, Fennema, Peterson, Chiang, & Loef, in press; Peterson, Carpenter, & Fennema, in press). In the third year of the project we conducted a series of case studies of six teachers focused on providing a deeper understanding of how teachers use knowledge about their students' thinking to build upon their students' prior knowledge and how their students' learning is influenced as a result.

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Research Basis for the Studies

All of the studies were based on research on the development of addition and subtraction concepts in young children. For reviews of this research see Carpenter, 1985; Carpenter and Moser, 1983; or Riley, Greeno and Heller, 1983; Verschaffel, 1989). Within this domain researchers have provided a highly structured analysis of the development of addition and subtraction concepts and skills as reflected in children's solutions of different types of word problems. In spite of differences in details and emphasis, researchers in this area have reported remarkably consistent findings across a number of studies and have drawn similar conclusions about how children solve different problems.

The research is based on a detailed analysis of the content domain. This scheme provides a highly principled analysis of problem types such that knowledge of a few general rules is sufficient to generate the complete range of problems. The taxonomy of problem types provides a framework to identify the processes that children are likely to use to solve different problems and to distinguish between problems in terms of their relative difficulty. The analysis is consistent with the way that children think about problems and solve them. Children initially solve word problems by directly representing the action or relationships in the problems. When children enter first grade most of them are able to solve a variety of problems, and the processes that initially are used to solve problems do not appear to have been learned through formal instruction. This informal knowledge provides a basis to give meaning to the formal operations and symbol systems that are taught in school, provided that children are able to make the connections between their informal knowledge and the formal mathematics of school.

The empirical research provides a fine grained analysis of the different strategies that children use and the major levels that they pass through in acquiring more advanced procedures for solving addition and subtraction problems. This knowledge provides teachers a basis to better understand children's thinking and match instruction to a given level of knowledge and skills.

Baseline Studies

In our initial studies, we gathered baseline data on first grade teachers' knowledge and beliefs about students learning of addition and subtraction and on the relation between teachers' knowledge and beliefs and their students' learning (Carpenter, Fennema, Peterson, &

Carey, 1988). We found that first-grade teachers were able to identify many of the critical distinctions between addition and subtraction word problems and the kinds of strategies that children use to solve such problems. However, teachers' knowledge was not organized into a coherent network that related distinctions between types of word problems to problem difficulty or children's strategies for solving the problems. In the same study, we found that teachers' knowledge of their own students' abilities was significantly positively correlated with student achievement on both computation and problem solving tests ($r = .32$ and $.31$ respectively).

In a related study (Peterson, Fennema, Carpenter, & Loef, in press b), we found a significant positive correlation between students' problem solving achievement and teachers' beliefs ($r = .32$). Teachers whose students were successful in problem solving tended to agree with the perspective that instruction should build upon children's existing knowledge and that the role of the teacher should be to help students to construct mathematical knowledge rather than to attempt to transmit it to them. These same cluster of beliefs were uncorrelated with students' achievement on a computation test.

The differences in the ways that teachers perceived their roles is illustrated in the following protocols. The first is from a teacher who had one of the highest scores on the beliefs scale; the second from a teacher who had one of the lowest.

> . . . like the teacher also has to be the learner. She has to pay attention to where the kids are, learn from them where they are, and dictate what her next step is because there are a lot of different learners and learner's styles. I guess learners must be actively involved in doing work and think about what they are doing and verbalizing what they are doing.
>
> It's a big role. I have taught first graders that when they first came in, they didn't have any concept of what adding was. . . The teacher has to do it step-by-step, and you have to explain it daily, you have to go over it until they start getting the concept. Subtraction was even harder for them to understand. It took a long time. . . You would have to verbalize it and talk through it almost every day, so they would start getting the concept.

The Experimental Study: Cognitively Guided Instruction

In the baseline studies we found that certain kinds of knowledge and beliefs of teachers were related to student achievement. The purpose of the experimental study was to determine whether we could affect positive changes in the relevant knowledge and beliefs of teachers and whether these changes would be translated into changes in classroom instruction and student achievement. We call our approach *Cognitively Guided Instruction*.

The teacher education component of Cognitively Guided Instruction (CGI) is based on the premise that the teaching-learning process in real classrooms, is too complex to be totally scripted in advance. As a consequence, teaching essentially is problem solving. Classroom instruction is mediated by teachers' thinking and decisions. Thus, researchers and educators can bring about significant changes in classroom practice by helping teachers to make informed decisions rather than by attempting to train them to perform in a specified way.

The staff development approach we have taken has been to help teachers understand children's thinking, give the teachers an opportunity to plan how to use this knowledge in their classroom, and to give them time to reflect on what happens as a result of using this knowledge. We have not developed instructional materials for teachers to use with children. Rather, we have watched as teachers have adapted their own instructional methodologies to include the use of their students' understandings. Teachers have either developed their own instructional materials or have adapted available materials.

There are direct parallels between the way we dealt with teachers and the vision we had of how they might use the knowledge we shared with them about their students. The baseline studies indicated that, like their students, teachers started with a great deal of relevant knowledge. We attempted to help the teachers build upon that knowledge and put structure on it. The process of teaching was perceived as problem solving (Carpenter, in press), and as such we did not provide ready made solutions for the problems of instruction.

The experimental study lasted one year and involved 40 first-grade teachers who were assigned randomly to an experimental or control group (Carpenter, Fennema, Peterson, Chiang, and Loef, in press). The experimental teachers participated in a 4 week workshop. The goal of the workshop was to help the teachers understand how addition and subtraction concepts develop in children and to provide them the opportunity to explore how they might use that knowledge for instruction. Teachers learned to classify problems, to identify the solution strategies that children use to solve different problems, and to relate these strategies to the levels and problems in which they are commonly used. This knowledge provided the framework for everything else that followed, and one and a half weeks of the four-week workshop was spent on it.

During the remainder of the workshop, teachers discussed principles of instruction that might be derived from the research and designed their own programs of instruction based upon those principles. Although instructional practices were not prescribed, the teachers discussed how to use the knowledge they had acquired in assessing their own students thinking and planning for instruction. Specific questions were identified for teachers to address in planning their instruction, but teachers were not told how they should resolve them. These questions included the following: 1) How should instruction build upon the informal and counting strategies that children use to solve simple word problems when they enter first grade? 2) Should specific strategies like counting on be taught explicitly? 3) How should symbols be linked to the informal knowledge of addition and subtraction that children exhibit in their modeling and counting solutions of word problems?

Results. During the following school year, we systematically observed in the experimental and control teachers' classrooms using a detailed observation scale. We measured the teachers' knowledge and beliefs, and their students learning. Although instructional practices were not specified, CGI teachers taught problem solving significantly more and number facts significantly less than did control teachers. CGI teachers encouraged students to use a variety of problem solving strategies, and they listened to processes their students used significantly more than did control teachers. As a consequence of allowing students to use different strategies during instruction and listening to students describe how they solved problems, CGI teachers knew more about individual students' problem-solving processes, and they believed that instruction should build upon students' existing knowledge more than did control teachers. Students in CGI classes exceeded students in control classes in number fact knowledge, problem solving, reported understanding, and reported confidence in their problem solving abilities. In spite of the fact that CGI teachers spent only about half as much time explicitly teaching number fact skills as control teachers, CGI students actually recalled number facts at a higher level than control students.

We also found that within the experimental group the teachers who were most influenced by the treatment had the highest levels of achievement. At the end of the year following the experimental treatment, experimental teachers' knowledge of their own students and their beliefs about learning and instruction were significantly correlated with students' problem solving performance ($r = .52$ and $.54$ respectively). The classes with the highest levels of adjusted achievement were those of teachers whose beliefs were most strongly based on the perspective that the teacher was not the ultimate source of knowledge and that instruction should be designed to help children construct solutions to problems for themselves. These teachers spent more time questioning and listening to their students and less time explaining procedures to them. They knew more about their students abilities, and their students had the highest levels of achievement (Peterson, Carpenter, & Fennema, in press).

Case Study Results

The experimental study demonstrated that we can change teachers by helping them to understand children's thinking and that those changes are reflected in what the teachers do in the classroom and in their students' learning. The following year we conducted a series of case studies of six teachers who had participated in the CGI workshop. The goal of the case studies was to better understand the nature of the changes we had found in the experimental study. Specifically the case studies were concerned with how teachers gained an understanding of their own students and how they used that knowledge to build upon their students' informal knowledge. In both cases the answers are relatively straightforward; however, in both cases it is not just what the teachers did that was critical; it was their understanding of their students thinking that allowed them to interpret students responses and modify questioning or instruction accordingly.

The critical element in the classes in which we observed the most impressive levels of problem solving was that the teachers were able to assess what their students were capable of so that they could continue to expand the students' knowledge by giving them increasingly challenging problems that were not beyond their capabilities. By listening to their students, these teachers learned that their students were capable of solving much more challenging problems than they previously had anticipated. They did not simply give increasingly difficult problems; they were able to match the problems to the students' abilities.

To assess their students, the teachers did not rely on written tests or formal assessment procedures. Instead assessment was an ongoing part of instruction. The teachers continually asked their students to describe the processes they had used to solve a given problem. The teachers almost never taught a lesson designed to teach specific procedures. In group discussions they asked students to explain how they had solved a particular problem, and students were encouraged to describe alternative solutions. Typically four or five different students would describe how they had solved a problem. In individual or small group work, the teachers also asked students to explain their work rather than showing them what they did wrong. Because the research the teachers had studied provided a coherent framework for organizing problems and the processes that children use to solve them, the teachers had a rationale for selecting problems and a context for interpreting the students' responses. Consequently they knew what questions to ask and what to listen for. They could attend to important variations in students responses and did not have to keep track of a vast array of unrelated details.

The teachers built upon students' informal knowledge by starting with problems that students could solve. Students spent a good deal of time solving and talking about these problems using a variety of informal counting and modeling procedures. The teachers allowed students to use procedures that were appropriate for them; and using their knowledge of problem difficulty, the teachers selected and adapted problems, so that different students were able to deal with problems that provided a reasonable level of challenge.

By talking about how they solved problems, children learned to reflect on how they solved problems and to articulate their solutions. The modeling and counting strategies that children used to solve different problems became more accessible so that they could readily be applied to a variety of problems and children could relate the strategies that they used for different problems. Once children's informal strategies were readily accessible and were objects of discussion, symbols were introduced as ways of representing knowledge that children already had.

Problem solving was not limited to solving word problems. A problem-solving orientation characterized the way students and teachers thought about any situation. Basic skills and concepts were approached from a problem-solving perspective. For example, children developed place value concepts out of problem contexts by extending their informal modeling and counting strategies. Children invented their own algorithms for solving multi-digit problems. Following is an example of one such solution:

> Well, 2 plus 1 is 3, so I know it's two hundred and one hundred, so now it's somewhere in the three hundreds. And then you have to add the tens on. And the tens are 4 and 7. . . well, um, if you started at 70, 80, 90, 100. Right? And that's four hundreds. So now you're already in the three hundreds because of that [100 + 200], but now you're in the four hundreds because of that [40 + 70]. But you've still got one more ten. So if you're doing it 300 plus 40 plus 70 you'd have four hundred and ten. But you're not doing that. So what you need to do then is add 6 more onto 10, which is 16. And then 8 more: 17, 18, 19, 20, 21, 22, 23, 24. So that's 124. I mean 424.

To understand how teachers build on children's thinking it may be informative to look at a situation in which a teacher missed an opportunity to do so. The example comes from a dissertation by Cheryl Lubinski, who is studying teachers' decision making. The teacher Ms J., participated in the CGI workshop, but obviously she has not assimilated the underlying principles of the program.

J. Wrote on the board

$$\begin{array}{r} 35 \\ \underline{+35} \end{array}$$

J. Now, who can add this for me? Adam.

A. 70

J. How did you get that?

A. Well I knew that 3 and 3 is 6, so 30 and 30 is 60. And 5 and 5 is 10, and 60 plus 10 is 70.

J. OK. You have the right answer; however, if I did 3 plus 3 is 6, and then I went to 5 plus 5 is 10, and I put that down, Adam, I'd have 610.

J. Wrote on the board

$$\begin{array}{r} 35 \\ \underline{+35} \\ 610 \end{array}$$

J. Is that the right answer?

A. No.

J. You have the right answer, but how could I do that to show it?

A. You could do 5 and 3.

J. Well, I can't. They live in different houses.

A. You add the fives and then you add the threes.

J. Well, I'm over here in the ones' house. What do I have to do? I'll bet, Linda, you remember [emphasis added] what I did. . .

In a stimulated recall interview following the class, Ms. J. made the following comment about this incident:

> . . He [Adam] had a very good way to explain it, but it wasn't explaining that I wanted him to carry the 10. . . You have so many children that will write down the 10 and then go to the tens' column and put down a number there too and come up with a three digit answer when it should be two. . . I thought his process, his thinking, was excellent; but he would not have been able to record it. . . He would have known it was wrong, but he wouldn't have known how to change it.

This episode illustrates the kind of thinking that students are capable of, and how we often miss the opportunity to capitalize on that thinking. We should not, however, be too critical of Ms. J. She is not just concerned with whether a student has the right answer. She asked Adam to explain how he got his answer, and she seemed to understand his explanation. She was still unwilling, however, to let go of her role as the dispenser of knowledge to try to build upon Adam's thinking. She was concerned that Adam's procedure would result in errors for him or for other students, however, the examples we have seen in other classrooms suggests that the concern is unfounded. We have not found that students' invented procedures are fraught with errors, and there is nothing in Adam's response to suggest that he could not have written his answer correctly.

Although Ms. J. does not yet embody all of the principles of CGI, she is, in fact, starting to listen to how her students solve problems. That is the first step. Many teachers took some time to adapt their teaching to the principles of CGI, and some only went so far. We are having teachers question some of their central beliefs of teachers, that they are the source of knowledge and that they have a responsibility to cover a specified amount of content. It takes time to change those attitudes. The workshop alone did not change the teachers; it was listening to their own students solve problems that made the greatest difference. The impact of the program and the change in teachers' behavior appears to be related to how carefully they listen to their children and to how much they believe that their children's thinking is an important determiner of what can be learned.

Perhaps the clearest example of the effects of listening to children comes from two of the teachers who changed dramatically during the year they participated in our case studies. At the beginning of the year they were most influenced by the details of CGI. They used the taxonomy of problem types to include a variety of problems during instruction, and they asked students to describe the processes used to solve different problems. But they did not really listen to the children or plan instruction to extend the children's knowledge. Their concern was with instruction and covering the material. If a child had difficulty explaining a solution, these teachers often would help the child in order to keep the lesson moving rather than attempt to understand the nature of the difficulty or encourage the child to clarify the explanation. At the same time these teachers felt compelled to cover all the content normally in their first-grade curriculum even though the children in their class had demonstrated that they already understood certain topics by the way they had solved related problems.

During the first third of the year as these two teachers reflected on what their children were telling them, they began to change. They began to really listen to their children; they began to give students time to explain their solutions, and they tried to understand unclear explanations. Concurrently they began to act on the basis of what they knew about their students. They no longer felt compelled to cover certain content; their concern became what their students knew and what they were capable of learning. They learned that their students knew a great deal more than they had given them credit for. They began to extend the content they taught to build upon that knowledge.

Critical Features of the Program

During the inservice program teachers acquired very explicit knowledge about children's thinking. It is an open question whether similar results could be obtained by giving teachers access to research based knowledge on children's thinking that focused on less specific aspects of problem solving, but we speculate that the specific knowledge about children's thinking in a clearly defined content domain was critical. The taxonomy of problems provided teachers a basis for deciding what questions to ask and the analysis of children's strategies gave them some insights for what to listen for. As one teacher commented: "I have always known that it was important to listen to kids, but before I never knew what questions to ask or what to listen for."

The information about problems and children's strategies had a face validity for the teachers; it made sense for them, so they were intrigued to see what their students could do. The problem and strategy analysis provided them enough structure so that the teachers had something to hang onto when they started. The analysis of children's solution processes is very robust; when the teachers started to talk to their students, the students' responses fit the patterns we had talked about. For the teachers listening to their own students was the critical factor. They began to see that their students were capable of more than they had anticipated, and that they, the teachers, did not have to explain everything.

Although our workshop focused on just a part of the mathematics curriculum, that knowledge provided the key for more far reaching changes. Once the teachers started listening to their students, they were able to extrapolate to place value, algorithms, and other topics in the mathematics curriculum; and for many of the teachers it affected how they listened to their students in other subject areas.

With respect to students' learning, there appeared to be several critical elements. Because the teachers had a better picture of what their students knew, they were better able to

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adapt instruction and provide problems at an appropriate level of challenge. For the most part, it meant giving students more difficult problems. The CGI focus on informal problem solving legitimized children's informal strategies and provided them with the perspective that what they learned in school should make sense and should be related to their real world knowledge and that they had the ability to figure things out for themselves.

Children were given the opportunity to construct problems and solutions that were meaningful to them. This appeared to help children to consolidate their informal knowledge and connect it to a variety of problem situations. By encouraging children to talk about how they solved problems, the strategies were made more overt and became objects of reflection for the children. The discussion of alternative strategies had another consequence. The children learned by listening to other children describe their strategies. In this regard the CGI classes share some common ground with the reciprocal teaching method of Palincsar and Brown (1984), except that explicit modeling of strategies is a less central characteristic of CGI classes.

In Conclusion

Whether CGI would work as effectively at other age levels and with other content is an open question. First-grade teachers seem to be especially open to listening to their students, and they may not be as constrained by the demands of content to be covered as teachers in later grades. Research knowledge in most other content domains is not yet as detailed or as robust as the knowledge about addition and subtraction. There also may be some unique characteristics about addition and subtraction that will not generalize well. Children's invented strategies tend to be additive, even when they are dealing with multiplicative situations.

Although many unanswered questions remain, our research to date suggest that giving teachers access to research based knowledge about students' thinking and problem solving can affect profoundly teacher' beliefs about learning and instruction, their classroom practices,their knowledge about their students, and most importantly their students' learning and beliefs.

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HARDIESSE ET RAISON DES RECHERCHES FRANÇAISES * EN DIDACTIQUE DES MATHEMATIQUES

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I - Deux cents ans après

Il y a un peu moins de deux cents ans en France, la Convention Nationale décidait que "la révolution qui s'était faite dans le système social et politique" devait "s'opérer aussi dans la théorie des sciences et des arts" (introduction aux Séances des Ecoles Normales, an IV, Imprimerie du Cercle Social) et qu'un de ses desseins était de "donner au Peuple français un système d'instruction digne de ses nouvelles destinées" (arrêté du 24 nivôse, l'an III de la République française une et indivisible). On sait que les scientifiques les plus prestigieux se sont alors attachés à développer des réflexions sur les meilleures méthodes d'enseignement pour initier le plus grand nombre d'élèves dans les sciences.

Je voudrais transmettre l'idée qu'il souffle un peu de cet esprit enthousiaste et hardi dans notre volonté de développer le champ de recherche que constitue la didactique des mathématiques en France et que notre démarche n'est pas exempte de cette tradition révolutionnaire. N'est ce pas aller contre les idées reçues que de chercher à décrire, à expliquer de façon rationnelle les phénomènes d'enseignement qui suscitent en général davantage l'empirisme ou l'opinion que le discours raisonné ?

A une date plus proche de nous dans le passé, à l'occasion de la première tenue en France de la conférence de notre groupe (en 1981, à Grenoble), G. Vergnaud introduisait les orientations théoriques et méthodologiques des recherches françaises en évoquant non leur caractère révolutionnaire mais leurs différences de fondements d'avec les recherches étrangères. Mon exposé esquissera l'évolution de ces recherches depuis cette date, en en présentant certes les aspects originaux mais aussi en essayant de les relier aux préoccupations qui animent les recherches étrangères car il me semble, et le travail effectué au sein du groupe PME y est pour beaucoup, que des liens objectifs se sont créés entre nos approches, ou du moins nos questions, et les approches et questions développées en dehors de la France.

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* An English version of this text will be distributed at the Conference.

Je ne pourrai évidemment présenter toutes les recherches françaises dans lesquelles est engagée une centaine de chercheurs. J'ai délibérément effectué un choix pour ne conserver que certains thèmes. Un des soucis largement partagés au sein de la communauté française est celui de l'établissement d'un cadre théorique original développant ses propres concepts et ses propres méthodes et satisfaisant à trois critères : pertinence par rapport aux phénomènes observables, exhaustivité par rapport à l'ensemble des phénomènes pertinents, consistance des concepts développés au sein du cadre théorique (Brousseau, 1986). Un large consensus se fait aussi sur l'exigence méthodologique d'avoir recours à l'expérimentation en interaction dialectique avec la théorie : le paradigme expérimental est conçu au sein d'un cadre théorique, les observations de l'expérimentation ensuite confrontées à ce cadre et susceptibles de le faire évoluer compte tenu des trois critères précédents. L'ambition d'un tel programme est manifeste et le respect de tels critères ne peut être le fait d'un seul chercheur ou d'une seule équipe, il ne peut être que le fruit d'interactions et de coopérations nationales. Plusieurs lieux institutionnels nationaux, où s'exercent de tels échanges, ont été mis en place en France : un groupement de recherche CNRS, un séminaire (3 fois par an), une école d'été de deux semaines (tous les deux ans).

II - Les rapports entre enseignement et apprentissage

Ce que nous appelons didactique des mathématiques en France recouvre l'étude des rapports entre enseignement et apprentissage dans leurs aspects qui sont spécifiques des mathématiques. Une idéologie très répandue suppose un lien de simple transfert de l'enseignement vers l'apprentissage: l'élève enregistre ce qui est communiqué par l'enseignement avec peut-être quelques pertes d'information. De nombreux travaux conduits au sein de PME ont fortement montré le caractère erroné de ce point de vue en mettant en particulier en évidence des caractéristiques de connaissances construites par les élèves à propos de notions arithmétiques, algébriques ou géométriques non contenues dans le discours de l'enseignement : ces connaissances sont locales, partielles voire erronées. Ces constatations laissent présager de la complexité des liens entre enseignement et apprentissage. Cette complexité est à l'origine et au cœur de nos recherches.

Dans une première période, les recherches françaises avaient entrepris l'étude de ces liens en se consacrant à l'apprentissage de savoirs mathématiques dont les processus liés au contenu disciplinaire étaient encore peu connus. Même si l'étude des leviers d'évolution des connaissances était alors déjà entreprise et la théorie des situations didactiques déjà développée (Brousseau, 1981), ces questions étaient abordées par les recherches françaises, en étant surtout centrées sur l'activité cognitive du sujet, en un mot en se restreignant au sous-système savoir-élève. Depuis huit ans les recherches françaises n'ont cessé d'accentuer leur orientation vers l'étude des dépendances entre enseignement et apprentissage, en cherchant à répondre à des questions du type : Comment caractériser les conditions à mettre en œuvre dans un enseignement pour permettre un apprentissage ayant des caractéristiques données a priori ? Quels sont les éléments d'une description d'un processus d'enseignement qui en assurent la reproductibilité du point de vue de l'apprentissage qu'il permet chez les élèves? De telles questions supposent que l'on ait choisi des critères pour caractériser un apprentissage, pour juger de la reproductibilité d'un processus d'enseignement.

Le critère fondamental qui conditionne notre approche de ces questions est celui du sens : Quel est le sens des savoirs que l'on désire faire acquérir ? Quel est le sens des connaissances construites par l'apprenant au cours d'un processus d'enseignement ?

Les recherches françaises en didactique ont manifesté la volonté d'appréhender globalement les situations d'enseignement, d'en élaborer une modélisation qui embrasse leurs dimensions épistémologiques, sociales, et cognitives et qui cherche à prendre en compte la complexité des interactions entre savoir, élèves et enseignants au sein du contexte particulier de la classe ou plus généralement d'une formation. Plusieurs approches de tels objectifs sont possibles. On peut ainsi vouloir repérer les "bonnes" situations d'enseignement et chercher à extraire les connaissances expertes d'enseignants "expérimentés" pour caractériser les valeurs des variables d'entrée (au niveau de l'enseignement) assurant un "bon" apprentissage chez l'élève à la sortie. L'approche choisie n'est pas celle-là, elle consiste plutôt à décrire le fonctionnement des situations d'enseignement comme celui d'un système dépendant de *choix* et soumis à des *contraintes* (*le système* *didactique*) à dégager ces contraintes et choix, et à repérer en quoi différents choix provoquent des apprentissages différents du point de vue du sens, c'est-à-dire la construction par les élèves de différentes significations des notions enseignées.

III - Choix et contraintes du système didactique

Un des axes des recherches en didactique a consisté à dégager les contraintes qui s'exercent sur le système didactique et à en analyser le fonctionnement. Les plus importantes d'entre elles sont

1 - les caractéristiques des savoirs à enseigner, en particulier des dépendances entre objets mathématiques qui doivent être prises en compte dans la constitution d'une cohérence du contenu destiné à être enseigné;

2 - les contraintes sociales et culturelles qui s'exercent au sein du projet éducatif dans la détermination des contenus d'enseignement

3 - les caractéristiques du temps de l'enseignement fixé par les programmes, en particulier sa linéarité

4 - les connaissances des élèves, leurs modes de développement cognitif qui conditionnent l'accès à de nouveaux savoirs

5 - la dissymétrie enseignant-apprenant par rapport au savoir dans les situations d'enseignement (*contrat didactique*)

6 - les connaissances des enseignants, leurs représentations à la fois, des savoirs mathématiques, de l'enseignement, de l'apprentissage et de leur propre classe.

Ces contraintes s'exercent de façon conjointe et n'ont été séparées que pour les besoins de l'exposition. Elles n'interviennent pas toutes aux mêmes niveaux du processus d'enseignement; ainsi les contraintes 1, 2, 3 et 4 affectent singulièrement la détermination des savoirs à enseigner (*transposition didactique*) en amont du processus d'enseignement tandis que les contraintes 4, 5 et 6 opèrent surtout en aval, dans le réalisation de l'enseignement. Les recherches françaises ont contribué à formuler ces contraintes dont l'étude soulève maintenant un intérêt plus large dans la communauté internationale. Si les termes de *transposition didactique* (Chevallard, 1985) et de *contrat didactique* (Brousseau, 1981) sont partis de France pour aller au-delà de nos frontières, les nécessités d'une analyse épistémologique ont aussi été ressenties en RFA ou sont actuellement affirmées dans des programmes de recherche comme le Research Agenda Project du National Council of Teachers of Mathematics aux USA; les règles implicites qui gèrent en classe les rapports entre enseignants et apprenants ont été l'objet de recherches de type ethnographique (par exemple, en RFA, la notion de *Arbeitsinterim*, Krummheuer 1983, présente des liens certains avec celle de contrat didactique), les représentations des enseignants constituent un paradigme de recherche en Grande Bretagne et aux USA (on pourra à ce propos consulter le rapport rédigé par Cooney et Grouws,1987).

Les choix possibles dans la constitution d'un processus d'enseignement s'effectuent en fonction des contraintes auxquelles est soumis ce dernier. Ces choix relèvent d'hypothèses en général implicites sur la manière de satisfaire aux contraintes du système, et la volonté de mettre en évidence l'existence de tels choix contribue à une meilleure connaissance des phénomènes d'enseignement dont les retombées sont très importantes pour une formation professionnelle des maîtres afin qu'ils puissent être à même de disposer d'outils explicites de prise de décision en classe, comme le remarque Romberg (cité par Cooney et Grouws, 1987).

Trois types de choix sont fondamentaux , ce sont les choix relatifs à

- la détermination des contenus à enseigner;
- l'aménagement des interactions entre apprenants et savoirs à apprendre;
- les interventions et le rôle de l'enseignant en situation de classe.

Les choix des contenus d'enseignement et de leur organisation reposent sur des hypothèses de type épistémologique et des hypothèses d'apprentissage (Arsac, 1989). Citons à titre d'exemple deux choix qui ont été prônés en France et mis à exécution à plus ou moins grande échelle. Les présentations axiomatiques linéaires (telle celle offerte par Choquet, 1964), issues du travail d'organisation fait par les mathématiciens au niveau du savoir savant ont été proposées pour la structuration des contenus d'enseignement et ont fortement influencé les programmes dits de Mathématiques Modernes. Elles reposent sur l'hypothèse que le sens des savoirs mathématiques provient de la structure logique hiérarchisée dans laquelle on peut les ranger; l'hypothèse d'apprentissage qui y est naturellement attachée est celle d'une accumulation de connaissances, une connaissance nouvelle venant s'ajouter aux anciennes sans les remettre en question ni même les modifier. Ces deux hypothèses permettent de satisfaire de façon très économique aux contraintes de cohérence du contenu d'enseignement et de la linéarité du temps d'enseignement.

En réaction à ce choix, l'association française des Professeurs de Mathématiques préconisait à partir de 1972 l'organisation des contenus d'enseignement en *noyaux-thèmes*, le noyau étant constitué "des notions fondamentales qu'au terme de l'année tout élève doit avoir acquises", les thèmes étant choisis par l'enseignant et les élèves, "soit pour motiver l'introduction des notions fondamentales, soit pour illustrer des utilisations de ces notions, soit encore pour nourrir des recherches supplémentaires dont la gratuité donnerait aux élèves un avant-goût des études libres que devenus adultes ils entreprendront peut-être." (Bulletin de l'APMEP n° 300, sept. 1975). Les thèmes potentiels sont innombrables -facture d'électricité, carré magique, marées, prix quotidien d'un chien en ville (*ibid.* p.437)-. Dans ces propositions, le sens des savoirs est fourni par ses contextes d'utilisation, l'apprentissage se faisant d'autant mieux que la même notion est rencontrée par les élèves dans un grand nombre d'activités (*ibid.*, p. 419). Une hypothèse sous-jacente est que l'élève pourra de lui-même abstraire les savoirs mathématiques des contextes dans lesquels il les a utilisés de façon plus ou moins implicite. En terme de dialectique outil-objet (Douady, 1985), l'aspect outil des notions mathématiques est privilégié; en conséquence, les contraintes de cohérence et de dépendances des savoirs et de linéarité du temps sont difficiles à satisfaire (ce qui explique probablement l'abandon rapide par les enseignants de manuels fondés sur le principe des noyaux-thèmes). La contrainte d'adaptation de l'enseignement aux élèves est remplie en tenant compte plus des goûts des élèves, de leur familiarité d'avec les contextes choisis que de leurs connaissances et de leurs capacités cognitives.

Différentes analyses de curricula existants et des hypothèses sous-jacentes relativement à un champ conceptuel donné ont été menées en préalable à des recherches sur l'enseignement -apprentissage des notions concernées.

Un autre type de choix s'effectue dans l'organisation des interactions entre élèves et savoirs à apprendre. Un choix extrême (et qui n'a pas cours en général) consiste à laisser l'interaction complètement à l'initiative de l'apprenant en ne lui présentant que le texte d'un savoir décontextualisé, d'un savoir sous une forme que l'on pourrait qualifier de *concept-definition* en reprenant le terme de Tall et Vinner (1981). L'hypothèse d'apprentissage sous-jacente postule l'appropriation du savoir par l'apprenant sans transformation de sa part, tel qu'il a été formulé par l'enseignant.

Dans les réalisations d'enseignement, il y a en général une contextualisation du savoir enseigné, c'est-à-dire l'organisation d'un contexte mettant en scène le savoir, contexte dans lequel s'exerce l'activité de l'élève. Les interactions entre savoirs et élèves se font par l'intermédiaire du contexte, du *milieu* (Brousseau, 1986). Les interactions prévues peuvent relever de différents choix.

Comparons ainsi trois situations introductrices à la notion de symétrie : dans la première, utilisée fréquemment dans les manuels, on fait plier une figure le long de sa droite de symétrie et on demande à l'élève de faire le plus grand nombre de remarques possibles; dans la deuxième, on lui demande de dessiner une droite telle que si l'on plie la figure le long de cette droite les deux parties de la figure se superposent, le papier sur lequel est tracée la figure pouvant être plié; la troisième situation ne se distingue de la deuxième que par l'impossibilité de plier le matériau rigide sur lequel est tracée la figure. Les processus cognitifs impliqués dans la réponse de l'élève à ces trois situations sont différents.

Dans la première, on sollicite une démarche inductive de l'élève (Johsua, 1987), on espère que de la simple observation il tirera *naturellement* les propriétés pertinentes du point de vue mathématique; dans les faits les critères de pertinence ne sont pas fournis par la situation, seul l'enseignant les connait. L'élève est en partie dégagé de la responsabilité de ses réponses et une partie de son travail consistera plus à chercher des indices de pertinence externes à la situation qu'à analyser la figure : il est en classe de mathématiques, il devra donc plutôt s'intéresser à des égalités, à des relations de parallélisme ou de perpendicularité qu'à la couleur de la figure par exemple. Cette situation doit nécessairement être conclue par l'enseignant qui donnera une liste de remarques. Une analyse théorique fine de ce type de situation est développée par Voigt (1985) qui souligne la fréquence de tels épisodes dans l'enseignement ordinaire.

Dans la deuxième situation, l'élève trouve dans la situation (en pliant) le moyen de savoir si sa réponse est correcte ou non, mais si le pliage est un instrument de validation, il est aussi un instrument de solution : l'élève peut avoir accès directement à la solution par essais successifs de pliage et ajustements; la question a été transformée, il ne s'agit plus que d'exercer une activité perceptive pour faire coïncider deux figures.

Dans la troisième situation, le contrôle perceptif est bloqué puisque l'élève ne peut plus plier le matériau; la question est bien de mettre en œuvre des propriétés de la symétrie pour tracer la droite de symétrie. Ce sont ces propriétés qui constituent l'instrument de solution. En revanche, la situation matérielle ne fournit plus d'instrument de validation de la réponse de l'élève et de ce fait ne lui permet pas de prendre conscience du caractère erroné de sa réponse (par exemple, dans le cas d'un rectangle s'il a tracé une diagonale) et de chercher à la modifier. La situation ne permet en elle-même guère l'évolution des réponses des élèves. D'une certaine manière, la situation ne peut être conclue que par l'enseignant comme dans la première situation.

Ces variations sur un exemple ont été choisies pour souligner que dans cette interaction organisée entre savoirs et élèves, deux critères jouent un rôle décisif quant à l'apprentissage visé : i) la question à laquelle les élèves répondent dans les faits; ii) les retours qu'offre la situation à la réponse de l'élève.

Les interactions entre la situation et l'élève peuvent provenir directement de l'environnement matériel comme dans le cas du pliage pour la symétrie, elles peuvent aussi tirer leur origine des connaissances mêmes de l'élève, la solution étant par exemple en contradiction avec ses connaissances antérieures. Les connaissances de l'élève lui servent de *critères de validité* de sa réponse (Margolinas, 1989). Au fur et à mesure de l'avancement dans l'apprentissage d'une notion mathématique, les situations peuvent faire davantage appel à des critères de validité et ne plus avoir recours à l'utilisation d'un milieu matériel. La même évolution se produit dans l'avancement dans la scolarité. Ce sont donc les connaissances construites par les élèves qui vont servir de leviers d'évolution de ces mêmes connaissances.

De nombreuses recherches françaises ont travaillé dans la conception de processus d'enseignement à la détermination de telles situations, les hypothèses épistémologique et d'apprentissage prises consistant à donner au problème une place centrale à la fois dans la signification des savoirs mathématiques et dans l'apprentissage de l'élève (Vergnaud, 1981). Les savoirs mathématiques tirent leur sens de l'ensemble des problèmes pour lesquels ils fournissent des instruments de solution; c'est confronté à des situations problématiques que l'élève engagera ses connaissances anciennes et si elles ne lui permettent pas de fournir une solution efficace, construira une réponse nouvelle adaptée au problème dans un processus interactif d'équilibres et désiquilibres. On reconnaît en toile de fond une hypothèse constructiviste d'apprentissage. Dans l'étude des rapports entre enseignement et apprentissage à propos de diverses notions mathématiques, une des tâches des didacticiens a donc été de concevoir des situations-problèmes préservant le sens du savoir à apprendre, dans lesquelles la connaissance à mettre en œuvre et à construire par l'élève, est entièrement justifiée par la situation (critère i), et qui permettent une interaction féconde avec ce dernier (critère ii). Encore faut-il que l'élève se saisisse du problème pour prendre en charge "seul" sa résolution, le lise et cherche une solution de la façon attendue. C'est à ce niveau que peut intervenir l'enseignant (cf. plus bas).

Les analyses théoriques fondant les constructions de situations sont validées dans la confrontation des phénomènes observés aux phénomènes attendus. Cela signifie que pour toute expérimentation de situations, une analyse théorique de toutes les conduites possibles des élèves confrontés à la situation est indispensable. C'est d'ailleurs cette analyse qui permet de donner un sens aux conduites observées des élèves, ces dernières prenant une signification par rapport à d'autres conduites qui auraient pu avoir lieu.

Une troisième catégorie de choix se situe dans la gestion en classe par l'enseignant du processus d'enseignement. C'est lui qui, par les informations qu'il donne et celles qu'il retient, permet ainsi qu'une situation-problème soit *dévolue* (Brousseau, 1986) aux élèves sans modification de sens pour ces derniers. De nombreux exemples de changement de signification opérés par les interventions de l'enseignant ont été relevés dans diverses études. Steinbring (1988) présente un cas de réduction de sens opérée par un enseignant qui, ayant constaté que ses élèves n'arrivent pas à résoudre le problème du calcul du périmètre d'un rectangle de dimensions variables x et y, segmente le problème en posant la question dans le cas où les dimensions sont des nombres concrets, puis demande une généralisation du résultat obtenu avec x et y. Le processus de résolution sollicité a été modifié par cette intervention : l'enseignant fait appel à une algorithmisation d'un procédé qui fonctionne sur les nombres au lieu de faire opérer sur les variables; le sens du concept de variable s'en trouve modifié : on ne lui accorde qu'une valeur de substitution au lieu d'une valeur d'objet. Cet exemple illustre bien le caractère paradoxal du contrat didactique : tout ce que l'enseignant entreprend pour faire produire par l'élève les comportements qu'il attend, tend à priver ce dernier des conditions nécessaires à la compréhension et à l'apprentissage de la notion visée (Brousseau, 1984).

Grenier (1988), dans l'étude d'un processus d'apprentissage de la symétrie orthogonale, analyse le cas d'une situation dans laquelle les élèves ont à dessiner la droite de symétrie d'un trapèze isocèle uniquement à l'aide d'une équerre et d'une règle non graduée. Les contraintes de la situation ont été choisies pour privilégier l'utilisation par les élèves de certaines propriétés de la droite de symétrie (incidence et orthogonalité) et bloquer l'usage de la propriété selon laquelle la droite de symétrie passe par les milieux des bases du trapèze. Mais les élèves détournent la règle de son usage habituel et soit la graduent, soit utilisent sa section comme instrument de mesure pour pouvoir construire les milieux, procédure qu'ils connaissent bien. L'enseignant constatant le changement de problème effectué par les élèves intervient pour poser à nouveau le problème initial et insiste à cette fin sur la précision que doit avoir la construction. Mais, alors que pour l'enseignant une construction utilisant des intersections et des orthogonalités est plus précise qu'une construction fondée sur des mesures approchées, le seul fait d'utiliser une mesure suffit pour les élèves à garantir le caractère précis de la construction; leur procédure de recours aux milieux s'est trouvée renforcée par l'intervention de l'enseignant.

Douady (1985) a souligné le rôle de l'enseignant dans l'institutionnalisation des savoirs c'est-à-dire la transformation en objet de savoir d'une connaissance qui n'a été investie par les élèves qu'en tant qu'outil de solution d'un problème . Une connaissance utilisée implicitement dans la résolution d'un problème n'est pas reconnue par les élèves, elle doit être dégagée par l'enseignant qui en explicite les aspects qui devront faire partie des savoirs appris et réutilisables ensuite. Dans un processus d'enseignement, l'enseignant se trouve donc aux deux extrémités des situations problèmes données aux élèves, il assure la dévolution du problème et l'institutionnalisation des savoirs en œuvre. Il joue donc un rôle primordial sur le sens des savoirs construits par les élèves. Si la théorie développée a mis en évidence ce rôle, les conditions et variables dont dépend son exercice sont moins bien connues. Elles commencent pour cette raison à être l'objet de recherches.

En effet, les recherches ont eu tendance à négliger le rôle de l'enseignant. Or les décalages entre ce que les analyses théoriques donnaient à prévoir du résultat de séquences d'enseignement et les effets observés ont amené des chercheurs à prendre pour objet d'étude le rôle de l'enseignant dans la classe. Ainsi Robert et Tenaud (1988) ont étudié l'incidence des interventions de l'enseignant lors de travaux de groupe d'élèves. Robert et Robinet (1989) ont cherché à analyser les représentations des enseignants sur les mathématiques et leurs enseignements qui ont une influence certaine sur les décisions spontanées de l'enseignant en classe et que la théorie pour l'instant ne prend guère en compte. Grenier (1988) a montré combien la marge de manœuvre de l'enseignant est liée aux conduites des élèves et à son interprétation des conduites des élèves.Les phases de bilan où l'enseignant fait le point sur les activités des élèves sont particulièrement sensibles de ce point de vue.

V - L'ordinateur en tant que constituant du milieu

Dans l'organisation des interactions avec les savoirs à apprendre, le rôle de l'ordinateur a été l'objet de plusieurs recherches ces dernières années. En effet, une analyse didactique est nécessitée par l'existence de nouvelles situations d'enseignement créées par la modification du milieu que constitue l'utilisation de l'ordinateur : recherches sur la nouvelle signification des concepts mathématiques médiatisés par l'ordinateur, recherches sur les nouvelles interactions permises entre savoirs et élèves, recherches sur le changement apporté au contrat didactique par l'usage de l'ordinateur, les rôles respectifs des acteurs élèves et enseignants se trouvant modifiés (Gras, 1987).

Soulignons cependant que le seul fait d'inclure l'ordinateur en classe n'implique pas une seule conception de l'enseignement et que des utilisations différentes sous entendant des hypothèses épistémologiques et d'apprentissage différentes sont possibles. Ainsi, une des possibilités originales de certains logiciels tels Cabri-géomètre, celle de production de représentations dynamiques d'objets mathématiques et donc de visualisation de propriétés mathématiques comme des invariants dans les transformations opérées à l'écran peut être exploitée en sollicitant des démarches différentes de la part de l'élève : induction de propriétés à partir de l'observation ou validation de conjectures par simulation à l'écran. Ainsi dans Cabri, le déplacement de figures construites par les élèves permet de disqualifier des constructions faites au jugé, le logiciel ne prenant en compte que les propriétés géométriques explicitement utilisées.

Plusieurs recherches ont été menées sur les interactions entre savoirs et élèves, en particulier sur la conception de situations jouant sur les caractéristiques de l'ordinateur pour permettre l'émergence chez l'élève de nouvelles stratégies de solution face à un problème et donc de nouvelles connaissances. A titre d'exemple, je citerai celle de Osta (1988) qui a conçu un processus d'apprentissage de la notion de système de référence dans l'espace à trois dimensions dans lequel l'évolution des stratégies des élèves est permise par les contraintes du logiciel utilisé, en l'occurrence Mac Space. Mac Space est un éditeur graphique conversationnel qui construit des représentations graphiques en perspective d'objets à trois dimensions, à partir de trois vues sur lesquelles peut agir l'utilisateur. C'est parce que l'exigence de communiquer explicitement à la machine les coordonnées des objets solides à représenter appelle la prise en compte de la structuration de l'espace particulière au logiciel que le problème à résoudre par les élèves est bien celui de la construction d'un système de référence. C'est la possibilité de voir à l'écran le résultat obtenu qui est facteur d'évolution des stratégies des élèves. Un résultat en perspective non satisfaisant perceptivement les incite à chercher la raison de l'erreur, cette raison amenant une autre représentation de la solution… La perception est utilisée ici en tant qu'instrument de validation et non en tant qu'instrument de solution, la complexité du système de référence de Mac Space et des objets donnés à construire empêchant que les ajustements par tâtonnement donnent une solution perceptivement satisfaisante.

Comme on le voit sur cet exemple, l'organisation de situations incluant l'ordinateur repose sur une analyse de même type que dans un environnement classique. L'analyse est peut-être rendue plus complexe de par le grand nombre de stratégies permises par certains logiciels, la complexité conceptuelle de ces derniers, et surtout de par la représentation que l'élève élabore du fonctionnement de l'ordinateur et qui évolue au cours de la tâche en interaction avec la représentation du problème par l'élève et ses démarches de solution. C'est un élément de complexité supplémentaire qu'il est impossible de ne pas prendre en compte si l'on veut attribuer une signification aux conduites des élèves et donc à la nature des apprentissages effectués.

V - Formes de fonctionnement du savoir

Brousseau (1981) a montré qu'à différentes organisations du milieu correspondent différentes formes de fonctionnement du savoir : certaines situations appellent des connaissances implicites en action, d'autres sollicitent de la part de l'élève l'explicitation des connaissances (situations de formulation) enfin une troisième catégorie de situations requiert la justification par les élèves de leurs explicitations (situations de validation). Les situations d'action ont été l'objet d'une attention particulière depuis longtemps et sont les mieux maîtrisées en particulier du point de vue des rétroactions du milieu. L'influence de l'école piagétienne est certaine quant à l'importance des préoccupations des chercheurs pour la question des rapports entre conceptualisation et action.

Plus récemment, les liens entre conceptualisation et formulation, entre conceptualisation et validation soulèvent un intérêt plus large qui est allé de pair avec l'apparition de recherches concernant des niveaux scolaires plus élevés (au delà de la scolarité obligatoire) qui exigent des niveaux de conceptualisation impliquant formulation et validation. Des recherches ont montré les liens dialectiques entre le statut cognitif des objets mathématiques chez les élèves et les formulations ou les validations mises en œuvre par les élèves à leur propos (Balacheff, 1988).

En particulier, les formulations usuelles du discours mathématique requièrent un certain niveau de connaissance des objets et relations mathématiques: ils doivent être suffisamment décontextualisés et détachés des actions des élèves. Des objets géométriques tels que points et segments doivent être suffisamment dégagés de leur contexte et perçus comme des objets indépendants pour recevoir un codage à l'aide de lettres par les élèves : ainsi un segment vu seulement comme bord d'une région, c'est-à-dire lié à une région, n'est pas codé par les élèves (Guillerault, Laborde, 1986). Inversement les exigences de fomulation conduisent l'individu à envisager autrement les objets mathématiques; en effet des problèmes spécifiques posés par l'activité de formulation obligent à de nouvelles analyses conceptuelles. Par exemple le vocabulaire adéquat manque, la complexité du discours à produire est trop grande parce que les relations envisagées entre objets sont trop complexes. La prise en compte de l'interlocuteur à qui est destiné le discours conduit à expliciter des données jusque là évidentes pour le locuteur, l'oblige à prendre conscience d'éléments implicites dans sa démarche, à opérer une distanciation et une décentration des objets afin qu'ils puissent être saisis par l'interlocuteur uniquement par des données verbales.

A contrario, la question se pose de la construction de situations d'enseignement permettant l'apprentissage de la formulation en mathématiques. En effet, les formulations spontanées des élèves en mathématiques contiennent souvent des implicites et des ambiguités. Usuellement le procédé pratiqué dans l'enseignement est la monstration, on demande à l'élève d'imiter le discours de l'enseignant ou du manuel. Des situations permettant une construction de formulations précises et non ambigues en réponse à un problème ont été conçues. Deux caractéristiques sont utilisées pour permettre une telle construction : la dimension sociale de l'activité langagière et sa finalité. La formulation de l'élève est destinée à un pair à qui elle est nécessaire pour réaliser une activité subséquente qu'il ne pourrait réaliser sans cette formulation. Le fait que ce soit un pair incite les élèves à veiller à la qualité de leur formulation pour que leur camarade réussisse l'activité qui en dépend. Un des problèmes non encore résolus de ce type de situation est que les retours de la situation s'effectuent par le camarade lecteur qui dans certains cas interprète des formulations erronées selon le sens voulu par l'élève producteur de formulations.

VI - Conclusion

Les recherches françaises sont diverses mais animées d'un esprit commun que j'ai cherché à communiquer dans cet exposé. Deux mots peuvent le résumer : hardiesse et raison, hardiesse de vouloir construire une rationalité des phénomènes aussi complexes que recouvrent les rapports entre enseignement et apprentissage. Les résultats acquis ces dernières années donnent cependant à penser que cette hardiesse n'est pas inconsidérée. Le débat scientifique, parfois vif, au sein de notre communauté contribue sans nul doute à cet avancement. Je souhaiterais qu'il continue de s'ouvrir à la communauté internationale pour y recevoir une validation.

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MATHEMATICAL LITERACY FOR ALL:EXPERIENCES AND PROBLEMS

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For the past 15 years I have been co-ordinating teaching innovations for compulsory education (which in Italy is the same for all pupils from 6 to 14 years of age).Our projects involve an high number of pupils (over 4000 this year,of mixed social-cultural extraction).In this lecture I will concentrate on the aims of these activities (mathematical literacy at the highest possible level for all pupils) and on some of the cultural and didactic choices made by us to reach these aims(in particular:construction of concepts and basic mathematical abilities through activities concerning the knowledge of important aspects of natural and social reality).These aims and choices are,on one hand,related to the celebrative framework in which PME-XIII is inserted,and on the other hand are related to the topics discussed in some lectures given during the recent PME meetings.

1.INTRODUCTION

From the start of our activities,the aims of mathematical literacy for all the pupils have been closely tied to the general educational aims of schooling .This has led to the construction and experimentation of a complete teaching project,for between 11 and 14 year olds,in mathematics and experimental sciences(in Italy,these subjects are taught by the same teacher),and then to a project for primary schools(6 to 11 year olds)which includes all the main subjects taught in Italy at this scholastic level.

The changes occuring in the extra-scholastic situation,the experience gained in the project experimentation, the difficulties met,the progressive refinement of our skills in analyzing the work in class and the ever increasing comparisons with international researches have induced us to modify the mathematical and cultural aims pursued and the methods of reaching them. Nevertheless some of the original characteristics(in particular,the allocation of the didactic work to all the pupils ,and the emphasis on teaching mathematics as an instrument of knowledge for important aspects of social and natural reality) have remained as they were initially.Today,these two characteristics form a pair,tied together not only as an ideological and cultural choice,but also on the basis of didactic considerations : in fact,in order to construct basic mathematical concepts and abilities in all the pupils it seems necessary to us to insert the learning aims in contexts rich in meaning and motivation for the pupils;plus,the "rationalization" processes concerning reality seem to develop linguistic skills and reasoning capacities needed to systemize and work the mathematical concepts constructed.

These considerations link the contents of my lecture to the contents of other lectures given at PME meetings during the last few years.In particular I refer to the Lesh PME-IX lecture (problem solving in "realistic" situations:(19)),and to the Carraher PME-XII lecture (development of important mathematical skills outside the school environment,in real life situations:(11)).The quoted researches justify the hypotheses(which also have similarities in psycholinguistics:see (15)) that the "context" in which the problem situations faced by the pupils are inserted does not only affect the resolutive strategies adopted by them,but can also influence their learning processes and their cognitive development.My lecture is dedicated to the deepening (on the basis of our experience) of the relationships that exist between the attainment of the aims of mathematical literacy (ambitious but necessary) that we propose to reach with all the pupils,and the cultural and didactic choices relating to the "context".

This lecture takes into account the research work carried out within our group and the discussions undertaken with other Italian researchers (in particular, M.Bartolini Bussi,P.Guidoni and M.A.Mariotti).The critical re- examination of the subject-matter and the draft version by E.Ferrero have been of precious help.

2.ON THE AIMS OF MATHEMATICAL LITERACY IN COMPULSORY EDUCATION

I would like to first mention the aims which I consider must be achieved by all the pupils during the compulsory education period with regard to the requirements for a non-subordinate insertion into society,and for participation in the cultural and social transformations taking place.

The list of aims takes into consideration:

- explorative work which some members of my group have been engaged on for years,examining in detail professional performances at various levels
- an analysis of the requirements for cultural and operational instruments for the choices in everyday life(both at "private" and "public"level)
- the preparations necessary to continue studying with success both in the schooling subsequent to compulsory education and in adult further education programmes. The study made by Bordieu (9) highlights certain selection mechanisms ,based on the social-cultural extraction,which operate at higher scholastic levels.At the lower levels,the school cannot normally close or reduce the gaps in the basic abilities(in particular logic-linguistic skills) amongst the pupils due to the different social-cultural origins .

The final aims,which in my opinion today justify a programme of mathematical literacy for compulsory education,can be expressed in general terms as follows(for details,see section 6):

A) mathematical modelling,such as the capacity to recognize and represent existing elementary quantitative relationships between variables of economic,physical and environmental interests , in deterministic and stochastic situations(from the arithmetical modelling level of a common word problem,to a level of formula modelling)

B) command of symbol systems governed by syntactic "rules",with the capacity to perform some elementary transformations based on these rules and to perform progressions to represented meanings and their consequent return(interpretation and transformation of simple formulae,substitution of values in formulae,etc.)

C) capacity to interpret(at least verbally) logic operations set out according to various rules,and to suggest(for simple significance problems) resolutive algorithms that can be handled by programmable electronic devices

D) capacity for mathematical reasoning(at least at recognition level of the validity,or less,of simple conjecture,and the management of simple precise deductions from data or hypotheses).

E) mastery of elementary mathematical vocabulary necessary for subsequent mathematical studies and for participation in basic science education.

These aims concern both the pragmatic dimension of mathematics and its cultural dimension as an instrument of knowledge at the service of the other sciences and as an object of knowledge.The necessary links for training in computer science are also taken into consideration.

With regard to the mentioned aims,in the past during our class work we encountered serious difficulties in obtaining acceptable percentages of final success.There follows a brief summary of these difficulties which seem to interact negatively,especially in the case of pupils from modest social-cultural environments .(Refer to (6) for a deeper analysis)

- linguistic deficiencies connected with the command of the spoken language as an instrument of thought.Above all,these affect the attainment of aims B),C) and D) [illegible] they do also influence the other aims.
- cultural factors concerning the conceptions that the pupils have about mathematics in general and certain mathematical contents in particular,plus those they have about the phenomena to be modelled
- metacognitive deficiencies concerning the capacity for thought on procedures and mathematical formalisms and the management of logical reasoning
- factors inherent to "didactic contract"(10)

Sections 3,4 and 5 will be devoted to the analysis of some theoretical tools and practical choices that we utilize to interpret and face these failure factors.

Another failure factor which is usually encountered in cases with pupils with learning difficulties or in "difficult" classes is the choice that many teachers make for the quality of the "intermediate"aims followed in these circumstances.The current temptation of the teachers is,in fact,to direct the work of the pupils with learning difficulties towards activities having little mathematical "value"(Chevallard) (use of stereotyped patterns for problem solving,application of calculation techniques,etc.).It is true that by doing this some bonuses are secured for the weaker students,but it is also true that the gap with respect to the other pupils inevitably increases as regards final aims.For this reason,we feel that a strict orientation of the "intermediate" aims towards the mentioned final aims is of prime importance.In practice,we engage pupils in independent problem solving activities (not guided by pre-established solution patterns) as early as the first years of the primary school; a growing importance is given to thoughts on the concepts and the mathematical procedures introduced during problem solving; we try also to gradually develop awareness about the different mathematical formalisms.In this lecture it is not possible to delve into greater detail regarding this aspect of our curriculum planning.A few examples can be found in section 6.

In our opinion,another choice necessary for correctly orientating the work of all the pupils towards the highest objectives is that to parallel the aims which must be reached at each moment with 100% of the pupils with aims which must be initially proposed to all the pupils,even without immediately obtaining from them high percentages of success and autonomy.

> For example,with regard to the writing of processes,after approximately one and half years of primary school we expect all the pupils to be capable of independently writing a linear sequence of the simple actions necessary for obtaining a certain "product"which has been soon produced in class.All the pupils however have already been "exposed" to more complicated writing tasks(writing of processes in which simultaneity,iteratives,checks,etc.are used). Another example is that of the gradual approach to the meanings of division:at the end of grade I (6 to 7 years) we are content if only a part of the class knows how to calculate "how many drawing sheets of 200 lire can be bought with 900 lire" and at the end of grade II (7 to 8 years) we are satisfied if only a part of the class knows how to divide 7000 lire amongst the 21 children in the class.These aims,however,must be achieved by all the children by the end of grade III.

This choice seems to be suitable for "forcing"(through immersion in a social context in which one talks of certain things,they are represented,they are seen "done") all the children to extend the "nearest development zone" (Vygotskij),necessary because under the guidance(individual, if necessary) of the teacher the independent mastering of the aim being pursued is achieved.

Finally,we note that the choice to insert the teaching of mathematics in activities regarding the knowledge of reality allows this continuous paralleling of the more advanced aims with aims that all the pupils are required to reach at a certain moment.In particular,at the starting of compulsory schooling,the children recognize,above all,the "real world feature" of the topics with which they are faced (for example:"calendar","buying and selling");this allows all the children to feel involved in the problems they have to face (even if only a part of the children are self-sufficient in mathematic resolution of the more difficult problems).

3.KNOWLEDGE OF REALITY AND THE LEARNING OF MATHEMATICS

The previous considerations lead us to the heart of the argument that I would like to develop in more detail in this lecture.This concerns the choice and didactic use of suitable "contexts"linked to the extrascholastic experience of the pupils as a decisive element in the mathematical literacy process, to overcome some obstacles and difficulties mentioned in the preceding section.

3.1 *Necessity of a specific theoretical framework*

The choice to "reconstruct" the contents and the sense of knowing mathematics with the pupils by means of teaching "by problems" well framed in "actual" topics motivating the pupils to the intellectual effort necessary,belongs to the tradition of educational innovations (activism) in Italy and in other countries.Our initial activities followed this stream. At the start of the 1980s,the difficulties that we had encountered forced us to reflect upon the theoretical tools made available by the research into the teaching of mathematics at international level.Our difficulties concerned the need to render the planning of the didactic situations regarding "reality" more flexible and productive within the group,the need to interpret correctly the achievements and the obstacles met by the teachers in class ,especially during the mathematization processes, the lack of instruments to communicate in a "transparent" manner our ideas and proposals ouside the group. Analyzing the existing theoretical tools ("re-framing" of mathemathical knowledge(12); "tool-object" dialectic(16),"conceptual fields"(25), "learning environment"(22),"realistic mathematics"(24)), I observed that the problem of the choice of context is taken into account only from the point of view of the specific mathematical learning aims,with a resulting "episodic" feature of the "contexts"in front of the organic unity of mathematical curriculum.In our experience,on the contrary,we ascertained the strong influence that the cultural quality and the organic unity of the "contexts" chosen may have both on general (logical, linguistic, metacognitive...) competencies and on strictly mathematical competencies.This called for an integration of the existing theoretical tools according to our experiences and our needs. Elements which I consider justification for expanding the theoretical frame in the organic unity sense of the "context" in which the mathematical knowledge is re-framed can also,in my opinion, be obtained from the history of culture (14),(21) and from the studies of ethnomathematics (11).

3.2.*Fields of experience,semantic fields and conceptual fields*

"FIELD OF EXPERIENCE" is a sector of the experience (actual or potential) of the pupils

identifiable by them,with specific characteristics which render it suitable (under the guidance of the teacher) for mathematical modelling activities,posing of mathematical problems,etc.Examples of fields of experience which we widely use in the projects for secondary school and/or primary school are as follows:"Machines","Commercial exchanges","Production in class","Sun and Earth","Genetics","The calendar", "Automation","The house in territory", "Natural numbers",... Each field of experience can be seen from three points of view:that of the teacher ("internal context" of the teacher),that of the pupil ("internal context" of the pupil: motivations,mental representations, etc.), and that of the elements making up the field of experience and of the exiting relationships between them ("external context").

These three points of view are useful for comparing and analyzing the didactic problems of the fields of experience in which the external context is at least in part attemptable by the pupil in "tangible" direct terms (as for "productions in class"), of the fields of experience in which the external context is attemptable by the pupil in intellectual terms (as happens with "natural numbers",with pupils who have a good practical mastery of them),and of the fields of experience in which access at an intellectual level to the external context is,for a large part,controlled by the teacher or from books(as in the case of "Genetics").

In the first three grades of the primary school,we prefer "fields of experience" of the first type,which are present anyway in our projects until the end of compulsory schooling.Amongst the possible fields of experience of the first type,for the first years of primary school we choose those for which it seems the "external context" can be better linked to the extrascholastic experience of the pupil and to the adult experiences,richer in links and relationships accessible to the pupils and meaningful from the logic-linguistic and mathematics point of view.A partial description of these choices will be given soon.

In general,programming didactic situations in a particular field of experience must lead to a "project" on how to make the internal context of the pupil evolve in time,making full use of the restraints of the external context to force the learning process towards the set aims.

All this gives rise to the problem of representation,which must be taken into consideration both for the transfer of relationships present in the internal context of the teacher and the pupil to the external context (under the form of identifiable "signs"),and for the development of the thought processes in the "internal context" of the pupil (in particular,the representation serves as an instrument of thought for relating the restraints deriving from the external context with the mental images).(Section 4 is dedicated to some aspects of the problem of representation).

In the end,we see that the fields of experience do not only evolve with respect to the representation which they form and which gradually transform into the internal context of the teacher and above all into the internal context of the pupil but,in certain cases,also for the historic change in the external context (one thinks of the field of experience of "Machines")

The fields of experience demonstrate a certain rigidity,due to their intrinsic nature of broad contextual references identifiable by teachers and pupils.From a didactic planning and analyses of the teaching-learning processes point of view,it can be useful to introduce a more analytic tool which,though safeguarding a unity of context,allows the substitution of one field of experience with another more suitable for the circumstances,without compromising the re-framing process of certain mathematical contents.For this reason,I propose the concept of "semantic field".

"SEMANTIC FIELD" is an aspect of human experience (real or potential)which appears to the researchers,in one or more fields of experience,as unitary,which cannot be broken down further,and which can be "rationalized" through pertinent,intense and meaningful use of

mathematical concepts and/or mathematical procedures.

A few examples of semantic fields present in our projects and which to me seem to possess,in a symbolic manner,the characteristics of the previous definition are: "purchases","calculation of product costs","routes covered on foot","balancing of levers","representation of the visible environment","time of the month","time of day", "history time","sun shadows"

The semantic fields must also be thought of as parts of the fields of experience in historic-evolutive terms,from three points of view:the "external context" point of view,the "internal context of the pupil"point of view and the"internal context of the teacher" point of view.

In my opinion,the link of semantic fields with education needs is assured by:

- theoretical evidence:in the a priori didactic analyses of many basic concepts of mathematics,one senses "meanings" which may be thought related to many fields of experience,joined together in common problem situations which then identify a semantic field.

For example,in the a priori analyses of concepts of angle,parallelism and direction,meanings are identified which may be referred back to the problem situations regarding the "sun's shadow"(a semantic field which can belong to different fields of experience:from that of the relationships between "Earth and Sun" ,to that of "Orientation").In the a priori analyses of the concepts related to the model $y = a + b/x$,references can be seen to problem situations inherent to the calculation of "production costs"(this semantic field can be inserted in the fields of experience of "class productions","industrial productions","commercial exchanges",etc.)

- empirical evidence:from our experience, the behaviours of the pupils often correspond to the didactic a priori analyses; ; common learning processes and strategies are observed in problem solving situations which can be associated to common semantic fields,even if inserted in different fields of experience

- historic-anthropological evidence:by analyzing studies of the history of science (21) and ethnology (1),many examples of common conceptual constructions and working strategies can be found built up in different fields of experience.In this case also,it is possible to find common semantic fields which gather together the problematic situations faced by different populations in different eras.

Having the concepts of field of experience,semantic field and conceptual field(Vergnaud,(25)) available,those concerned with didactic planning and those carrying out didactic research can utilize work instruments of some benefit to:

- plan flexible didactic innovations,set out according to a double curriculum (based on "fields of experience"identifiable by teachers and pupils,and on "conceptual fields" justified by current disciplinary systems and cognitive processes analyses) which productively and intensely interacts in the "semantic fields".

- analyze,in particular,the capacities and difficulties that are met in the didactic practice of the reality-discipline interlacing.

With these theoretical instruments available,we are reconsidering the structure of our projects for the primary schools and comprehensive schools,gradually locating deficiencies and inconsistencies and evaluating the implied efficiency of the choices made in the past.

To give an idea of the present structure of our projects,I will make a few examples:at the start of the project for the primary schools,at 6-7 years , the field of experience of "commercial exchanges" and the conceptual field of "additive structures" are interlaced in the semantic field of "purchases" (about 150 hours of work in the classroom)(2) .The same conceptual field of the "additive structures" is also interlaced with the field of experience of "calendar",in the semantic field of "time of the month"(about 80 hours) and in the semantic field of "weather"(about 70 hours).Then,at 7-8 years,the field of experience of "class productions" start to interlace in the semantic field of "calculation of production costs"with the conceptual field of "multiplicative structures" and the conceptual field of the "additive structures" .At the end of the project for the compulsory school,a 13-14 years,we have [illegible] examples of interlacing,such as that betweeen the field of experience of "machines" and the conceptual field of "multiplicative structure" in the semantic field of "balancing of levers"(about 30 hours).

3.3.*Teaching-learning problems linked to fields of experience concerning "reality"*

When referring to our work in the primary and comprehensive schools,it seems that the choice of suitable fields of experience relating to natural and social extra-scholastic reality offers the following opportunities for mathematical literacy in compulsory education:

- *clear-cut use of the external context for conceptualizing* ;we are gradually becoming aware of the need(with regard to the analyses of the processes of learning that are recorded in class,and with regard to the management of the problem situations by the teacher) to distinguish betweeen the different types of problem situations proposed to the pupils in our fields of experience.For example,there are situations in which the adaption of the pupil to the restraints of the external context is natural enough,and the duty of the teacher becomes relatively easy;these can involve both aspects of natural reality and aspects of social reality.

An example of the first case concerns the geometric modelling of the relationship between the length of a shadow and the altitude of the sun : starting with the comparison between the spontaneous ideas of the pupils (the majority of whom initially are convinced that,in the morning,the sun's shadow is longer at 11 o'clock than at 9 o'clock,because the sun is stronger at 11 o'clock),then followed(if necessary) by observation and direct measurement,most of the 9 to 10 year olds construct,in an irreversible manner,the "lower the sun-longer the shadow" model which can be gradually developed with the aid of the teacher into a synthetic geometric representation .An example of the second case is the emergence of distributive strategies in payment problems of the type:"What is the total cost of four pencils,each of which costs 320 lire ?" The nature of the "structured material" of money (external context) and the experience of use of money already acquired by the children (internal context) prompts them to separate the 300 lire and the 20 lire according to:
320x4=(300+20)x4=300x4+20x4

Having started in our group (7) to analyze the reasons why certain problem solving situations stimulate in the children themselves an effective and sufficiently spontaneous adaption,these seem to differ according to whether one is dealing with adaption tied to perception(therefore immediate compliance with the restraints of the external context);or to conceptions and habits present in the social context (usually acquired through the common language and behaviours) which are prompted by the problem situation set;or to thought mechanisms which seem to occur "spontaneously" in the interior context of pupils (for example,when a 7 year old is mentally solving a problem of "change", a nett preference for completion strategies is observed,the same one used throughout for "giving change" in shops,even if the children haven't yet experienced buying in shops!)

There are however problem solving situations in which the work of concept construction and modelling requires the teacher to guide the work against the obstacles that originate from the common language,from widespread cultural concepts(4),from the most spontaneous perceptions that one has for a certain situation,and maybe even from primitive models rooted in the mind.

We have various examples to illustrate this:in the analyses of the shadow phenomenon,the 9 year old children realise that (at the same hour) the longest nail casts the longest shadow,but the majority think that when changing nail,it is the difference between the length of the shadow and the length of the nail that stay constant ,not the ratio (probably a primitive additive model comes into play,or an immediate interpretation in additive terms of the verbal expression "if the length of the nail increases, then the length of the shadow increases").Another example ,linked to the previous one,is that of the analyses of the "balance condition of a lever".It is not easy for a 13 year old to arrive at modelling in terms of "constant product amongst the applied force and the distance of the point of application from the fulcrum".In spite of the non-homogeneity of the dimensions,many youngsters put forward the model " F + D = Constant" (it is possible that this derives from the fact that,at a verbal level,the expression "as the force increases the distance decreases" initially suggests an additive model).Another situation which must be carefully handled by the teacher is that of probability modelling (according to Mendel's laws) of transfer of the hereditary characters .In this case,obstacles arise caused by the lack of a directly accessible external context that can be reasonably interpreted,as well as by concepts of common meaning on inheritability.

You may ask,would it not be better to avoid those types of situations now considered to be somewhat ineffective for conceptualizing mathematics.In reality these are compulsory stages (significant even from a historic point of view) if one wants the pupils to gradually understand that there are levels of "intuitive evidence" and "intuitive " ways of thinking which must be exceeded if a rational and working command of certain phenomena is to be reached,and that mathematics may have an important role in this passage from intuition to rationalization.

- development of linguistic competences in rationalization (bases of verbal language) and modelling (bases of algebraic language) activities. The considerations up to this point are directly linked to the problem of the development of linguistic competences regarding the various languages important for mathematical activities.The work in suitable fields of experience allows the pupils to be "forced" to use the different languages in a precise and pertinent manner,and to develop the verbal competences linked to mathematical reasoning and the construction and mastery of algorithms.Referring back to (8) and (17) for further investigation into this aspect,I would just like to briefly mention a few examples.(See section 4 for further considerations).

The first concerns (at a syntactic level) the field of experience of "class productions".In the individual recording of the production of a plum-cake,some children write that"you break the eggs and then you separate the whites from the yolks".The immediate performing of these "instructions" shows(immediately from the clear facts) the inadequacy of the linguistic expression used and the opportunity to pass to different expressive forms.Another example concerns (at a lexical level) the use of the term "direction" with the meaning of "common property of parallel straight lines" or "a dot on the horizon",overcoming the ambiguities that "position" leads to when it refers to needles of compasses,shadows,spirit levels,in the work in the semantic field of the "sun's shadows").

-development of correct and productive metacognitive attitudes: : these concern,in particular,the recognition of the necessity to make hypotheses,the comparison between the mathematical model put forward and the modelled reality,the comparison between different models or different risolutive procedures,the analyses and comparison of different forms of mathematical representation.The presence of an objective material external context,linked to the normal experiences of life,offers (when the child still does not have sufficient experience in mathematical contexts) extraordinary opportunities.

For example,let's consider the phenomenon of the shadows,and the problem situation in which the range of shadow directions of a nail needs to be completed ,a few shadows only having been traced (for example,the directions of the shadows at 10 o'clock and 11 o'clock need to be traced,the shadows at 9 o'clock and 12 o'clock are traced).The child can refer to a complete range recorded on other days (but then he must make the assumption that the angles through which the shadows revolve are of the same amplitude) ,or consider the day's shadows and assume that the shadows revolve through equal angles in equal time.In any case,these assumptions need verifying,and this in turn leads to a refinement of the measurement and recording methods.Then comes the problem of semi-darkness...on the subject of which it is still possible to make assumptions and models,and then try to obtain the instruments for their verification...

The complexity of the "real" problem situations allows the development of the capacity for hypothetical reasoning and modelling through the inescapable restraints and obstacles present in the external context of the semantic field.This complexity of "real" semantic fields,which at first may appear to prevent formation in the mathematics field,is very important to develop competencies at linguistic and metacognitive level (above all with pupils who have not had the opportunity to develop them in their social-cultural environment).(See also section 5).

4.LINGUISTIC ASPECTS OF MATHEMATICAL LITERACY

In our experience,for several years(at least until 1981) we have tended to overlook the linguistic aspects of mathematical literacy,trying from time to time to adopt those linguistic representations which seemed to allow the pupils a more direct access to mathematical concepts and procedures(in particular,in the activities of mathematization).We gradually became aware that this underestimation of the linguistic aspects and the "naive" didactic choices made in the linguistic sphere,produced a very weighty selection for the weaker pupils(in particular,for the pupils with a poorer social-cultural background).The planning of complex strategies,the explanation of concepts,the comparison between different strategies and deductions were almost barred to the pupils who did not know how to handle the hypotheses,comparisons etc. using the verbal language.We therefore started to dedicate an increasing amount of attention to the development of the verbal language as an "instrument of thought".We then realized that a more detailed examination of other aspects of the verbal language and other languages used in mathematics was required,in particular with regard to the algebraic language.I will now review some aspects and difficulties inherent to the development of linguistic abilities which,from our experience,we feel relevant ,at the same time limiting myself to the verbal and algebraic languages.These problems have been widely studied at an international level in recent years (in particular,I refer to the work of C.Laborde(18)).I will attempt to highlight some points which I feel require further investigations .

VERBAL LANGUAGE at present seems to be the heart,not only (obviously) of the expression of concepts and procedures and mathematical reasoning,deductions and thoughts,but also of the heuristic aspects and planning of the resolutive strategies for mathematical problems. Thanks to the extensive"practice" of writing which we provide for in the primary schools starting from grade I,we have realized (in terms that we are starting to clarify:(17)) the critical functions that the hypothetic reasoning seems to have in problem solving,as an instrument of construction and management of a fictitious "space-time" in which the pupil groups together the alternatives to explore,makes attempts,restructures the initial problematic situation gradually acquiring new information...The numerous records that have been accumulated seem to indicate a clear "following each other" between the emergence of these forms of reasoning and the effective use of verbal language in problem solving ,and specific activities previously performed in particular fields of experience(especially "Productions in class", "Sun and Earth", "Machines").

Of course,the evolution towards more sophisticated and specific verbal language forms (such as those of mathematical "deduction"),starting with the verbal forms of expression developed in the fields of experience,must be guided by the teacher.However,it seems that if the children possess the level of factual deduction it is possible (also by acting on metacognitive aspects:thoughts on reasoning methods,and using past experiences in the fictitious space-time of hypothetic-heuristic reasoning) to reach the level of mathematical deduction (including "reductio ad absurdum" !).This does not seem at all possible with pupils incapable of efficient verbal handling of the hypotheses at factual level or with those not having the quick-wittedness necessary for mentally moving around in "unrealizable" settings (20).

ALGEBRAIC LANGUAGE : in relation to the different functions that it assumes in internal mathematics and in the work of mathematization,and in relation to our experiences,it seems that the difficulties and obstacles that the weaker pupils encounter need to be investigated in greater detail,and the suitable didactic strategies for overcoming them found.

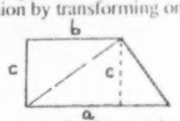
- *stenographic function*: :it is apparently taken for granted and widely used as common practice.Various difficulties and risks are concealed in it which have strong repercussions on the learning levels and attitudes of the pupils towards mathematics,both at the start of,and following,compulsory education.We think that algebraic notations are in general introduced into the school too early,and that their dogmatic introduction as stenography of verbal expression can generate obstacles.

Considering the first point,it is well known that in the arithmetic problem solving situations many pupils concentrate their attention on the choice of the sign (and not of the meaning !)of the operation to perform on the numerical data ;this sign subsequently activates the working of the calculation for producing the answers.We have verified that by greatly delaying the introduction of the signs (by which time many meanings of the operation to be "put in signs" have been acquired) and setting,in the context of our fields of experience, arithmetic "realistic" (19) problems without explicit numerical data (with the request to "indicate which data is needed",and to "put forward the resolutive strategy in general"), the final percentages of success in the solving of word problems can be varied considerably (see section 6).

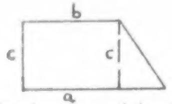
With regard to the second point,the following aspects seem particularly important.In the first place,the fact that many numerical expressions and many formulae are not consistent when expressed in verbal terms (the most natural order in which the calculations are verbally expressed is different to the order in which the algebraic expression is written).In the second place,the fact that while in the algebraic language a letter keeps its meaning throughout subsequent transformations of the "formula context" in which it is inserted,in the verbal language,the value of a certain word (for example,"number") is defined by the linguistic context in which the word is inserted: let's say "the sum of two consecutive odd numbers is obtained by adding to the first number,which can be represented by an even number plus one,the same even number increased by three" and we write :E + 1 + E + 3 . "E" does not only mean "even",but first means "an even",the second time it means "the same even".The clarification of these aspects (following(18)) has allowed us to understand some of the difficulties that pupils regularly encounter in relating verbal statements and algebraic representations (for example,the tendency for pupils to use the same letter in the same context with different meanings); it has also allowed us to plan specific thought activities on this argument with pupils.

- *transformation function* : both in the mathematics field and in mathematization activities the transformation of a formula into another according to the algebraic transformation rules offers some decisive advantages; this situation,however,is not easy to handle at didactic level.One of the advantages is the possibility to determine,using purely formal means ,the equivalence of different formulae.

For example,this occurs in the mathematical sphere when the construction of the formula to calculate the area of a right-angled trapezium is suggested.From two different decompositions,two formulae are obtained,and these must produce the same numerical result (when substituting "numbers" to "letters") as they concern the same figure.However ,it is interesting to induce the pupils to reflect on the fact that it is possible to reach the same conclusion by transforming one formula into the other:



$$\frac{a\cdot c}{2}+\frac{b\cdot c}{2}\rightarrow bc+\frac{(a-b)\cdot c}{2}$$



The same can also happen in modelling activities.For example,when determining that for the calculation of the cost of a product unit,two different formulae can be used (linked to two different ways of thinking of "unit cost ":as "total cost divided by the number of product units",or as "cost incorporated in the single unit of the product+ fixed cost divided by the number of product units"): $U = (n.C + F)/n$ $U = C + F/n$

Another advantage derives from the possibility of "discovering"(in the mathematization activities,or in the internal mathematical activities) relationships between the variables which are neither "evident" in the original verbal expression,nor in its transformation into formula.

As an example,consider the following problem : "A person lends a sum of money $C(0)$ with compound interest i (annually).How much money should he expect after n years ? " From the formula $C(n) = C(n-1) + i.C(n-1)$ we can obtain by simple transformations: $C(n)=C(o).(1+i)^n$

We may summarize in the following diagrams the mental operations connected with the "transformation function" :

$$\begin{tikzcd}
\text{FORM 1} \arrow[r] & \text{FORM 2} \arrow[d] \\
\text{SEM 1} \arrow[u] \arrow[r] & \text{SEM 2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{FORM 1} \arrow[r] \arrow[d] & \text{FORM 2} \\
\text{SEM 1} \arrow[r] & \text{SEM 2} \arrow[u]
\end{tikzcd}$$

For the pupils,the "commutativity" of these diagrams is not entirely evident .It seems that many of the difficulties encountered are at a metacognitive level.However,the gradual conquest of this form of awareness seems without doubt necessary for the aim of mathematical literacy (in its turn rigidly connected to necessities deriving from the growing computerization of many human activities.Some of these difficulties examined are already often encountered in the use of standard professional software).

5. "TOOLS OF KNOWLEDGE" AND "MATHEMATICAL OBIECTS"

The work carried out in the fields of experience largely results in the construction of concepts and procedures which,however,often remain at an implicit operative level;then the pupil is not able to recognize them , transfer them to other fields, connect them with other concepts or meanings.The problem of how to "arrange" the concepts in a consistent and explicit manner for the pupils has often been discussed:the two most common ways in Italian schools (and,possibly,abroad) are,on the one hand,the theoretical arrangement according to traditional patterns and,on the other,the theoretical arrangement according to more recent and rigorous points of view.Teaching research has emphasized the need to carry out a systematic work on the mathematical "objects" constructed at an implicit operative level(as "tools "(16) of knowledge),in order to "de-contestualize"(12) and "institutionalize"(10) them.Moreover,teaching research has often emphasized the need to (re)-construct arrangements of mathematical knowledge in pupils,according to their need to do so,rather than to impose them.

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Considering the teaching and cultural choices made in our projects,we have also to face the need to deal,relatively autonomously (with respect to fields of experience related to "actual" reality),with basic mathematical issues,which cannot be brought about in a natural and direct way,as "instruments" of knowledge of reality (e.g. natural number properties,operations with fractions,geometrical conjectures and demonstrations of theorems such as the theorem of Pythagoras,etc.),even though they are based on concepts -such as the concept of number,or the concept of the right-angle triangle-largely and directly used in "real "situations.

In our project for comprehensive school,approssimately 40% of the work on mathematics is presently devoted to the arrangement of the concepts introduced in the fields of experience and to the internal developments of mathematical knowledge.In order to plan this work consistently,we have been taking advantage of the opportunities offered by the history of mathematics,although not in the traditional way.In fact,we do not provide data on the history of mathematics as a "distraction" form,or a further enrichment of the usual work on mathematics,nor do we introduce work situations which attempt to recreate(how successfully ?) the debate on certain problems (e.g. incommensurability) of certain historical periods.Instead,we try to closely interrelate a statement of facts,changes,etc. on the historical evolution of certain mathematical arguments with pupil activities,in which they are asked to compare the mathematical activities "before"and "after" a certain innovation or discovery,the advantages of one formalism with respect to another,common language and mathematical language and so on.

This approach has a number of important technical and cultural aims (connected,directly or indirectly,with the aims specified at the beginning of the lecture and with that of overcoming the difficulty which makes their fulfilment on a large scale impossible):

- to convey an idea of mathematics as a result of the historical evolution of mankind,promoted dialectically by the need to know and master reality and by the internal need for exactness,consistency,etc.;so we hope to change the cultural concepts of the pupils (and related environment) that often hinder the learning process by diverting it to marginal objectives;
- to make the pupils aware of the crucial issues of the work on elementary mathematics- in particular,the variety of possible conventions and formalisms-
- to carry out technical activities on some internal aspects of mathematics,within a cultural framework,thus explaining their significance also in connection with their applications

The approach we have briefly outlined above involves serious problems concerning both linguistic and metacognitive competences;as we mentioned before,it is highly selective for those pupils who cannot master linguistically comparison,implication and hypotheses ,nor can they "decentralize" the concepts used or formalisms produced,and so on.Notwithstanding the difficulties met,we feel that the the work in the "fields of experience",which sets the basic concepts and procedures necessary for the work on mathematics ,can also prove very useful in constructing the linguistic and metacognitive competences needed(see section 3.3).

So far,we have examined the work we have carried out to stimulate reflection on mathematical objects and procedures in comprehensive school (11-14 year olds).With a view to a gradual and consistent work starting at the beginning of primary school,a reflection on the mathematical "objects" should also be carried out in primary school;by way of example,I would like to mention two of these activities:

-age group 7-8:after the pupils have learnt how to perform payments using coins in a number of ways,they are asked to compare all the different ways of paying,for instance,100 lira,using

Italian coins, and all the ways of breaking down 100 as a sum of two or more numbers...
- age group 8-9: children are asked to compare the calculation strategies produced by them trying to solve division problems of various types,within the fields of experience of the project(e.g.to divide a 23000 lire cost betweeen the 19 pupils in the class; to find out how many sheets,21 cm wide,are needed to make a paper strip 450 cm long); the main aim is that of gradually reaching (being aware of),by selecting the more economical and general strategies,a universal algorithm for the written calculation of division.This activity - related to various researches carried out in other countries (23) (24) - is very important ,since not only does it lead all the pupils to reason and learn a written calculation technique (page 193 in (23)),but also accustoms them to a significant performance from a metacognitive point of view,and,furthermore,seems to bring about a deeper and more uniform mastering of the different meanings of division (for further details on this research,see(8)).

6.RESULTS SO FAR OBTAINED AND RELATED PROBLEMS

A reliable assessment of the results obtained by adopting the teaching and cultural approaches we have mentioned before is not easy:as often happens in these circumstances,it is difficult to assess how much the results obtained depend on these choices and how much they depend on other choices which we make and are not aware of,or on the quality and attitude of the teachers who volunteered to experiment these projects.

Vis à vis,I would like to point out the work conditions of the teachers who experiment the projects:they must take part in at least one meeting a week (average length 2-3 hours),plus extra meetings in September and June (at the beginning and end of the school year).The meetings are usually organized dividing the 200 teachers of the team according to the age group of the pupils or according to the the topics to be discussed,and are co-ordinated by university researchers or teachers-researchers.In these meetings,teaching suggestions and materials proposed by teacher-researchers mixed groups are examined and discussed ,the learning difficulties recorded in class are analyzed and interpreted,proposals from other meetings are compared and documentation and research materials from outside are examined.

Another difficulty in realiably assessing the results depends on the assessment methods used:the main risk is to mix up "learning results reached in depth and permanently" with the results of mere adaptation to superficially acquired schemes,activated through the "format" of the tests to which the pupils are subjected,or through a terminology that recalls standard strategies,etc..In view of this,we have been trying to refine the methods and means for assessing learning results;we have also tried to set a wider range of learning assessing methods: not only standard tests with open answers (which all pupils taking part in our projects undergo from 2 to 4-5 times a year),but also "analyses of episodes" (accurate recording of whatever happens in the classes,"observing" work sessions over 2 hours long),and "case analysis"(study of individual pupils,particularly at a very low level,for very long periods,e.g. 2-4 years).The entire set of these learning assessment instruments used in our projects has allowed us to provide the following general information on the results we have reached,which would be interesting to compare with different socio-cultural realities and different teaching approaches:

with the primary school project, by working on all the main subjects taught in Italian primary schools(with particular reference to linguistic education):

i) at the end of the first year,we can have 90% of the pupils at a "primary alphabetization" level,consisting of the ability,on the part of the pupil,to manage additive composition,additive breaking down,and completion problems (up to the level: "I would like to buy something that costs 900 lira; I have 400 lira,how much more do I need ?),to read the temperature on a thermometer,to complete a sequence of numbers with 2-3 consecutive blanks(up to 30) (for example:"fill in the blanks:11,12,.......,16,.......,19,20); the same children should also be able to produce a written text telling in a comprehensible way "what was done at school during the morning",and to comprehend a simple descriptive or narrative text;

ii) the percentage of success(with reference to the same skills) virtually reaches 100% during the second year (it should be noted that in Italy failing the first year is rare;in the region where I work,less than 2% of the pupils fail in their first year).Thus ,only children with serious handicaps affecting their mental functions never reach primary alphabetization (brain-damaged,Down Syndrome children and serious cases of autism); in Italy these children are placed in ordinary classes with other children(with specialized teachers dedicated to them)

iii)we can,by the end of the 2nd year,have 80% of the pupils solve measurement-division problems,manage their first sharing-division problems ,neatly write the operations required to carry out a common activity,write precisely and represent graphically a route,measure and compare heights(in cm) and represent such heights in scale on graph paper;

iv) by the end of primary school,95% of our pupils reach a level by which they can solve problems in which they are asked to divide a given sum of money (e.g. 85000 lira) between 23 pupils,deciding the operation for themselves and carrying it out;measure two segments drawn on paper,assessing the difference in length between them; measure angles and drawn angles of a given measurement;represent on a "blank ruler" a decimal number; make a graph (on paper graded in millimetres) relevant to a table of data,choosing the suitable unit of measurement for the axes;"read" distance in scale; put into written words a production activity,with checks and choices...It is interesting to note that this success rate does not vary much between classes of middle- high or low social extraction;

v) again at the end of primary school,80% of the pupils can write a general procedure on the grounds of some particular examples worked by them (e.g. a procedure for the calculation of the area of an irregular plane figure by triangulation);solve a word problem requiring a chain of operations (with no intermediate back up questions);put into words a procedure for ordering two names ; draw a model for the logical behaviour of the key x on a calculator respecting priority of operations.It should be noted that virtually all children in Italy reach the end of primary school and that in our classes the percentage of children who are made to repeat one year,even due to long absence,does not exceed 3%.

With the *comprehensive school project* ,by the end of compulsory education (in Italy 14) we have the following results:

vi) betweeen 90 and 95% of the pupils can carry out the activities described in item iv)

vii) approximately 75% of the pupils are able to solve,autonomously, problems requiring mathematization through formulae and/or equations for a physical or economic situation (aim A,section 2);to find out whether a geometrical or number property applies only to some numbers(or figures) and to identify cases in which it does not apply (D);to recognize and explain the difference in meaning of the sign = in the following statements BASIC: X= X + 1 IF X=1 THEN 130(B);to make an algorythm to sum the cubes of the first 50 odd numbers (C) It should be noted that in the classes where our teachers work,approximately 15% of the pupils fail at least one of the three years and approximately 5% of the first year pupils do not finish comprehensive school;thus,the success rate mentioned above goes down to 70% if the whole population of pupils starting the first year is considered.

The data indicated in vi) are apparently contradictory with those mentioned in iv):the percentage of success in the same type of problem seems to be higher at the end of primary school rather than at the end of comprehensive school;this is because the pupils starting our project in comprehensive school are at much lower level than the pupils who finish our experimental primary school ,and pupils can not always catch up in the three years of comprehensive school.The above data also show that the more qualified goals (in view of the aims indicated in section 2) reached by the end of comprehensive school,(when the project is carried out only during comprehensive school) account for satisfactory results only in 7 cases out of 10.It is difficult to assess whether this depends on:

- the fact that it is actually impossible for the pupils to reach the more ambitious learning targets(because of the genetic and environmental situation of the weaker pupils)
- the fact that it is difficult, with children over 11,to work on the basic linguistic-verbal skills which are of tantamount importance in reaching such targets;

- the fact that the teacher of mathematics works with the class only 6 hours a week(out of a total of 30 hours/week),which is not enough to work effectively with the weaker pupils in the fields related to the teaching contract,to linguistic and logic skills and to metacognitive aspects,which are often dealt with in a contradictory way by the other teachers.

Possibly,each of these three aspects might play a significant role;indeed,the targets we have considered are difficult to reach,although we do feel that the pupils who end our primary school project are closer to these aims than the pupils who finish our secondary school project.Finally,we have noticed that-the socio-cultural extraction of the pupils being the same-where the methodological-teaching approach is along the same lines as the mother tongue teacher (who also teaches history and geography and works with the class 11 hours per week)the learning results are remarkably higher by the end of the three-year comprehensive school project.

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RE-EXPLORING FAMILIAR CONCEPTS WITH A NEW REPRESENTATION

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This paper is a preliminary report on how did adults with solid mathematical background re-examine familiar concepts in the light of a representation unknown to them. They explored the concepts related to the notion of linear functions using the Parallel Axes Representation within a computer-based environment developed especially for that purpose. We briefly describe the representation and the environment, we analyze cognitive episodes observed during our subjects' work and discuss potential implications of the use of new representations to re-examine concepts in mathematics.

Introduction

"Thinking is hard work. Once we have understood a mathematical process, it is a great advantage if we can run through it on subsequent occasions without having to repeat every time (even though with greater fluency) the conceptual activities involved. If we are to make progress in mathematics it is, indeed, essential that the elementary processes become automatic, thus freeing our attention to concentrate on the new ideas which are being learnt - which in turn must also become automatic." (Skemp, 1987, p. 61)

This "meaningful automation" process is one of the powerful tools of mathematics learning (and mathematics creation), which was noted by many mathematicians before the Psychology of Mathematics became an object of explicit study. Alfred North Whitehead, for example, discusses at length the "enormous importance of a good notation" and the nature of symbolism in mathematics: "...by the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain. It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle -- they are strictly limited in number, they require fresh horses, and must only be made at decisive moments." (Whitehead, 1911, p. 59)

This process is closely related to the human capacity to represent and encode mental objects, and work within representational systems, and is indeed a necessary condition for significant progress with new subject matter.

On the other hand, Freudenthal (1983, p. 469) writes: "I have observed, not only with other people but also with myself ... that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question." Following Freudenthal, we suggest that there may be some advantages to undoing the automation process, namely to direct our attention to reviewing the "source of our insights" which somewhere in the past were incorporated to our mathematical background as automatisms. Using Freudenthal's words, our purpose would be to engage in an "unclogging" exercise. One example of such an exercise dealing with the concept of variable is described in Arcavi & Schoenfeld (1988). In this paper we present preliminary results of another "unclogging" exercise, this time concerned with the notion of function, and particularly with the notion of linear function.

The exercise consisted of the re-examination of function-related concepts in the light of a representation unknown to them. Our purpose was to observe how do competent students and adults think and investigate when they are exposed to a representation of functions with which they were unfamiliar, although the concepts were known to them.

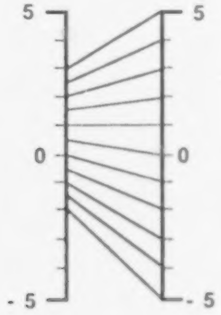
The notion of function is usually learned and "mastered" via two main representations: the algebraic and the Cartesian system. A student who has learned and mastered the notion of linear function will know, among other things, what slope and intercept points are, how to manipulate and operate with them, and how they are represented algebraically and graphically. A detailed description of the knowledge framework for linear functions, which includes both the concepts the connections among them, is provided in Schoenfeld, Arcavi & Smith (in preparation).

In this paper, we first describe the representation used in our exercise, the Parallel Axes Representation for functions, and the microcomputer-based environment designed for its exploration. Then we present and discuss episodes that occurred when the representation was investigated by competent adults and students. Finally, we suggest cognitive and educational implications for discussion.

Parallel Axes Representation (PAR)

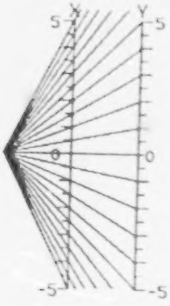
The Parallel Axes Representation (PAR) consists of two parallel vertical axes (number lines). The axis on the left is used to represent the domain of the function and the one on the right is used to represent the co-domain. The mapping of a number (pre-image) in the domain to its image in the co-domain is represented by the segment that joins them, called the *mapping line* . A mapping line in PAR encapsulates the same information as the Cartesian point, namely it represents the pre-image--image pair (or in the Cartesian terminology the abscissa-ordinate pair). Thus, a function, which is represented in the Cartesian plane by a set of points (in general a curve), takes the form of a bundle of mapping lines in PAR.

The following figure shows the representation of **f(x)=2x-1**. By picking a specific mapping line and following its trajectory one can read from the graph, for example, that 0 (pre-image) is mapped into -1 (image), 3 is mapped into 5, etc.

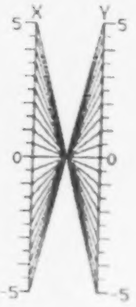


It becomes apparent that some of the concept names linked to the notion of linear function are strongly loaded with Cartesian connotations. For example, "slope". The name suggests precisely what we visually perceive in the Cartesian Graph: a certain measure of the inclination of the line with respect to the x-axis. Whereas in PAR, slope -- $[f(x_2) - f(x_1)]/[x_2 - x_1]$ or alternatively the **m** in **f(x)=mx+b** -- is reflected in the co-domain as the "expansion" or "contraction" (combined with "reversal" in the case of negative **m**) of an interval in the domain.

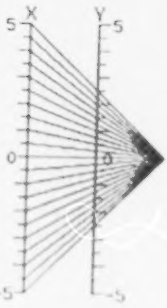
PAR also provides with a further opportunity to visualize the notion of slope: the mapping lines of a given function (extended where necessary) intersect in one and only one point (except when **m**=1). This point we call the *focus*. The location of the focus with respect to the axes is related to the value of **m**. Whenever **m**>1, the focus will lie to the left of the x-axis, if 0<**m**<1 the focus will lie to the right of the y-axis, and for **m**<0 the focus will lie between the two axes. The following figure illustrate the focus location for some members of the family **f(x)=mx**.



f(x)=2x



f(x)=-x



f(x)=0.5x

Since each focus defines one and only one linear function, we can represent in PAR a linear function by means of the focus alone. This feature of the representation enables to view a function as one single entity, and operate with it as such.

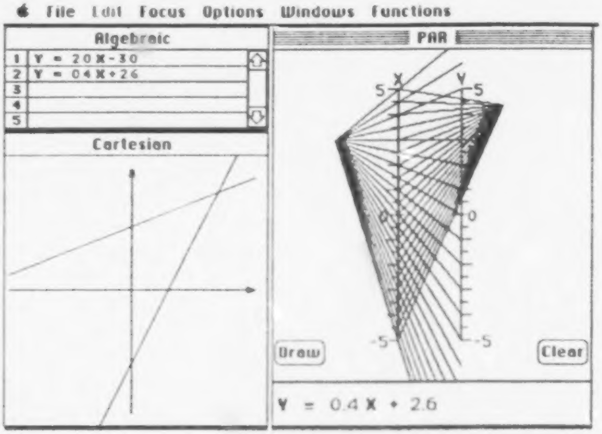
Another idea which has a very different appearance in PAR is the notion of "fixed point" of a function [the pair (**x**,**x**)]. Whereas in the Cartesian system it is not visually salient, in PAR, a "fixed point" (we should rather call it a "*fixed mapping line* ") is represented by an horizontal mapping line, perpendicular to both axes.

PAR may also support the visualization of composition of functions. By adding a third parallel axis, and by letting the intermediate axis act as both co-domain of the first function and domain of the second, one can visualize two functions and their composition.

These characteristics make PAR potentially rich as a medium for our "unclogging" exercise in which, by virtue of the exploration of the representation, one has to unpack certain automatisms developed during the experience of using the Cartesian plane. In the process, some "taken for granted" issues have to be re-examined.

The computerized environment

The computerized environment, designed for the purpose of exploring PAR, was programmed in Lightspeed Pascal for the Macintosh. It provides users with a tool for computations and graphing, which relieves them from the burden of tedious and time-consuming work. The computerized environment has the capability of graphing a function given in its algebraic form, in PAR and in the Cartesian system simultaneously. It allows for juxtaposition of the representations; superimposition of graphs; on-line change and control over certain graphical features (scaling, precision up to two decimal digits, drawing speed, etc.). In the environment, linear functions can be graphed with their focus by extending the mapping lines when necessary. In addition, there is an option to hide the segment bundle and show only the focus. A typical PAR screen is shown in the following figure.



1 Y = 20X-30
2 Y = 04X+26
3
4
5

Y = 0.4X + 2.6

A more detailed description of the environment is available in Nachmias and Arcavi (1988). A copy of the software is available from the authors upon request.

The study

For the purpose of this study we invited adults and advanced high school students with solid mathematical background to participate in the "unclogging" exercise. In this exercise we requested them to investigate how the focus location is related to the parameters **m** and **b** in **$f(x)=mx+b$**. Although the general goal was stated, they had the freedom to choose any approach they wanted, taking advantage of the features of the computerized environment.

Each of the students worked alone during 2-6 hours in separate sessions. Our approach for observing their work was naturalistic. We requested the students to think aloud and to share with us their immediate goals, their expectations and the way they settled their surprises whenever their predictions were different from the computer feedback. Two observers and a tape-recorder were used to document the sessions. Occasionally the subjects were asked questions to clarify what they were doing.

Analysis and discussion of selected episodes

In the following we present two episodes which occurred during our subjects' investigation of linear functions in PAR.

1 - RA is a bright 16 year old high school student in the 11th grade. He takes mathematics and physics as a major. He is highly motivated, and when he was told about the experiment he gladly accepted to take part. He was then introduced to PAR, and was asked to investigate it freely. He started by looking at linear functions with negative **m** and, after about 10 minutes, he "discovered" the focus, namely that all the mapping lines intersect in one point. He concluded that foci are located either in between the axes (when **m** is negative) or to the left of the x-axis (when **m** is positive). At this point he was asked whether he saw any possibility for the focus to lie to the right of the y- axis. His immediate reaction was "no", probably because he thought that he already exhausted all the cases. Without any prompt, he decided to think more carefully about that.

His reasoning proceeded as follows: "If we are to have a focus to the right of the y-axis, a mapping line from say 3, will have to end up in a "lower" number. Do we have any lines of that sort? Let's look at the Cartesian. If 3 goes to a smaller positive number, we will have a line which goes up to the right but below the diagonal [y=x] in the first quadrant, therefore, oh, that's right, we will have lines with slope between 0 and 1".

This analysis of the Cartesian system led him to conclude that for 0< **m**<1, the foci will be to the right of the y-axis. Then he noticed that there are "regions" in the PAR system which have "same sizes" [left to the x-axis and right to the y-axis] but correspond to foci locations for different "domain sizes" of **m** [(1, ∞) and (0,1) respectively]. He then decided to go back to the Cartesian system in order to look for traces of that puzzling "asymmetry". After some thinking and experimentation he realized how the same "asymmetry" is viewed in the Cartesian system. According to him he never thought of that before. At the end of the session, he spontaneously said: "the work with PAR, helped me to put together loose ends in the knowledge I had".

In this episode RA used the computerized environment in order to explore characteristics of the new representation on the basis of the known representation. In his exploration he relied very much on the translation between the two graphical representations. First, the translation helped him to discover the unknown properties of PAR and then it supported a more careful look at some properties of the known Cartesian system that went unnoticed during his previous instruction, and which probably did not hinder his performance in standard Cartesian problems.

2 - RO is a graduate student in instructional technology. She has a B.Sc. in general science. She explored PAR during four sessions with a total time of about 6 hours. She was very thoughtful regarding the questions she posed to herself for exploration. Whenever her predictions were at variance with the computer graph, she patiently stopped to reflect about the source of the discordance. At a certain

point in her exploration she correctly summarized all the possible locations of the focus as a function of the parameter **m** in **f(x)=mx+b**. When she referred to the case where **m**=0 in which the function y=0x+1 appeared on the screen, she correctly concluded that the focus will be located on the y-axis at 1. In order to assess to what extent her conclusion was general for any value of **b**, she was asked to predict the focus location for y=-3. We did not pay much attention to the difference in format between the computer would have displayed the function (y=0x-3) and the way we actually posed our question (y=-3). Therefore we were surprised when the following dialogue took place:

Q: What do you think we will obtain for y=-3?
RO: It's not a function ... without x, right? This will be a point in PAR. [Then she immediately drew correctly the Cartesian graph of y=-3] ... In the Cartesian for every x we will have -3. When y=-3 we will have a parallel [to the x-axis] line in the Cartesian, which means that it's not a point [in PAR]. ... Wait ... but it does not depend on x, in fact what the **b** does ... is to bring all the mapping lines to **b**.
Q: So [what is the representation] for y=-3 ... ?
RO: It's a point.
Q: Where?
RO: I'm trying to think what is a function... ... In the Cartesian we will have a parallel line... ... For every x we will have y=-3, that is to say that all the mapping lines will meet at y=-3. ... Now I'm trying to think... The first time you asked, I said a point then I said a line [in the Cartesian] now again I think it is a point...

This episode was indeed ephemeral, and very quickly RO was able to analyze her momentary confusion and overcome it. However, it is an interesting illustration of the "unclogging" role played by PAR. RO, knew that, in PAR, y=0x+b would look like a segment bundle with their focus on the y-axis at **b**. She also knew that y=-3 was a line parallel to the x-axis in the Cartesian representation. However, when working with PAR, she first related y=-3 to a point in the number line representing the y-axis, rather than to a function. The source of her confusion was the possible double meaning of y=-3 (a constant function and the ordinate of a point). The underlying double meaning of y=-3 came to the surface because it was reinforced both by particular features of PAR and by the context. PAR highlights that y=-3 can be regarded both as one element of the co-domain as well as a constant function. Moreover, RO was working with the "focus only" option, according to which a Cartesian line is represented as a single point in the PAR window. She was not confused when she was working with the form y=0x+b in the algebraic window, however had to struggle for a while to realize that the source of her confusion was the double meaning of the form y=-3.

After using symbols such as y=-3 in the Cartesian and algebraic representation, for many years, RO was not explicitly aware that they may carry different but related meanings. However, while working with an unfamiliar representation, she needed to "unclog" her conceptions and to re-examine the similarities between the represented meanings in the light of their differences.

Initial conclusions

The above episodes, and several more which we intend to present at the conference, share one characteristic: the exploration of PAR brought to the fore issues which were underlying people's well established knowledge and automatisms within a certain knowledge domain.

These findings raise several questions, that we believe, need to be addressed in both theoretical and empirical studies related to mathematics learning:

- The role of a representation of a mathematical idea seems to go beyond the mere goal of having a tool to handle that idea. Could it not be that by introducing a new representation, we are not only establishing a way to express an idea or a concept, but also re-examining and consequently learning "more" about those ideas and concepts?
- If new forms of representation have the benefit of "unclogging" certain ideas, and thus increasing our knowledge, does it make sense to talk about knowledge in all or nothing terms? What exactly does it mean to "know" a mathematical concept? If by redirecting our attention to "known" concepts we are able to "know more" about them, it would make sense to talk about partial and incremental stages of knowledge, rather than in terms of a zero-one situation.

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LA CONSTRUCTION DU CONCEPT DE FIGURE CHEZ LES ELEVES DE 12 ANS

(Gilbert Arsac, Université Lyon 1, 43 bvd du 11 Nov 1918, 69622, Villeurbanne CEDEX, France)

Abstract :Geometry is generally the field choosen to initiate students to the use of reasoning. In France, for instance, it occurs for twelfth children. This learning leads to an overthrow of the preceeding school use which is that geometry is essentially the art of making accurate drawing : now, it will be necessary to argue about figures and it is no more possible to be satisfied with mere ascertainement. If we take a constructivist hypothesis for learning, we must therefore find situations where mere "reading" on the figure leads to a failure from the point of view of the child himself. Such a situation is presented here, and we give results of experimentation in classes.

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1)Problématique

Le programme national de mathématiques de l'enseignement français comporte en classe de cinquième (c'est-à-dire pour les élèves de 12 ans) une "initiation au raisonnement déductif" et les commentaires des programmes privilégient la géométrie comme domaine de cette initiation. Or, en géométrie, l'appel au raisonnement suppose la reconnaissance de l'insuffisance de la constatation sur le dessin, c'est-à-dire le renversement de l'usage établi jusque là dans les études de l'enfant. Dans les classes précédentes, l'enjeu du travail de l'élève en géométrie était de tracer des figures précises, et de constater par simple observation ou par une mesure à l'aide d'un instrument un certain nombre de propriétés. Si l'on veut amener l'élève à démontrer, ce qui sera l'un des objectifs des études de l'année suivante, il faut donc au préalable avoir levé l'obstacle de l'évidence du dessin. Si l'on élimine l'appel à l'autorité de l'enseignant imposant dans la classe une nouvelle règle suivant laquelle on ne doit pas uniquement faire appel à la figure, et si l'on se place dans une perspective constructiviste, on doit donc chercher à bâtir des situations dans lesquelles l'appel à la figure conduise à un échec du point de vue de l'élève.

Un premier but de la recherche présentée ici est de construire un exemple d'une telle situation et de vérifier par une expérimentation dans les classes si cette situation provoque bien la mise en échec des conceptions relatives au rôle du dessin que nous supposons présentes initialement chez les élèves. Un deuxième but est de montrer quelle conception de la figure peut alors remplacer chez les élèves leur conception initiale et d'examiner expérimentalement si cette conception se met bien en place ou s'il y a stagnation, voire même régression, étant donné la mise en échec de la conception initiale, chez les élèves. Ceci nécessite un minimum de mise au point théorique sur le concept de figure auquel nous procédons maintenant.

Nous distinguerons dans la suite le dessin de la figure, désignant par dessin le dessin concrètement tracé sur une feuille de papier (ou dans le sable pour Archimède) et par figure l'objet mathématique dont le dessin n'est qu'une représentation.. Ainsi, la figure est un élément du "monde mathématique" et non du monde sensible. L'exemple archétypique du carré et de sa diagonale, laquelle est commensurable au côté sur le dessin, mais non sur la figure, rappelle que cette appartenance au monde mathématique lui confère des propriétés éventuellement contradictoires avec la lecture ou la mesure du dessin. C'est que les constituants de la figure (points, segments) peuvent être considérés comme ayant un statut d'objets idéaux (point de vue grec) ou comme simplement définis par des axiomes (suivant un point de vue moderne). Pour notre propos, comme les considérations précédentes ne peuvent pas être exposées à un enfant de douze ans, il nous suffira de retenir les conclusions : la mise en place de la notion de figure comportera toujours un rejet de l'expérience sensible du dessin au profit d'une conception plus élaborée de la figure. Ce rejet peut être constaté sur des exemples beaucoup plus simples que celui de la diagonale du carré : ainsi, l'impossibilité pratique de dessiner une tangente à un cercle n'ayant qu'un point commun avec le cercle conduisait Protagoras (Dumont 1988, fragment B7 de Protagoras) à rejeter la géométrie parce qu'elle est contradictoire avec le témoignage de nos sens pour lesquels la tangente et le cercle se confondent sur une certaine longueur. Il est plus difficile de définir positivement quelle conception de la figure peut être proposée à des enfants de douze ans. L'expérience semble toutefois montrer que la figure peut être considérée à cet âge comme un dessin "qui serait infiniment précis" ce qui permet de rendre compte à la fois de deux constatations contradictoires : l'échec de l'appel à la précision de la figure et la possibilité d'améliorer presque sans fin cette même précision.

2) Choix d'un problème

Nous sommes partis de la constatation suivante : lorsqu'on pose aux élèves le problème "de l'inégalité triangulaire" sous la forme suivante (ou une forme proche):

Choisis trois nombres a, b, c, est-il toujours possible de trouver un triangle dont les mesures des côtés soient ces trois nombres ?

la plupart arrivent à trouver, grâce à des essais sur plusieurs dessins, l'inégalité du triangle c'est-à-dire l'idée que le plus grand côté doit être inférieur à la somme des deux autres, mais en général ils n'arrivent pas à décider si l'inégalité doit être stricte ou non. Ainsi dans cette situation, la difficulté à conclure à partir du dessin apparaît naturellement aux élèves et non comme une question artificielle soulevée par l'enseignant. Cependant l'expérience montre aussi que cette question est noyée parmi toutes celles que peut soulever le problème et n'apparaît pas forcément comme primordiale aux yeux des élèves. Pour permettre que le débat se centre sur la question qui nous intéresse, nous avons donc décidé de fabriquer un énoncé amenant directement sur la question qui met en cause l'appel au dessin. Il s'agit de l'énoncé suivant :

Existe-t-il un triangle dont les côtés mesurent 5cm, 9cm et 4 cm ?

Cet énoncé concentre effectivement la question sur le cas qui nous intéresse, ce qui amène d'ailleurs à éliminer le cas général de l'inégalité triangulaire. En imposant une unité de mesure et des mesures précises des côtés, il permet d'assurer que tous les élèves auront le même dessin, et donc que l'impossibilité de conclure à l'aide des seules mesures devrait apparaître plus paradoxale encore.

3) Situation expérimentale, hypothèses.

Bien que ce problème ait été proposé à des élèves travaillant hors classe, nous relaterons ici ce qui s'est passé en classe : ce problème a été expérimenté dans 8 classes où les élèves travaillaient par groupes de quatre et dans plusieurs d'entre elles ces groupes ont été observés et enregistrés au magnétophone par des observateurs. Les résultats obtenus ont ensuite été analysés. En définitive, l'observation porte sur 10 groupes d'élèves dont 8 en situation de classe dont il sera principalement question ici. La situation de classe utilisée comporte une organisation très précise que l'on peut trouver en Arsac (1988) ou Balacheff (1988). Nous n'en donnons ici que les étapes essentielles :

-dans un premier temps (une heure environ), les élèves recherchent le problème par groupes de quatre ; pendant ce temps, l'enseignant n'intervient pas, en tous cas pas sur le contenu du problème. Ce premier temps s'achève par la production collective d'une affiche dans chaque groupe, comportant une réponse à la question posée et une explication destinée à convaincre les autres groupes de l'exactitude de cette réponse.

-dans un deuxième temps (une heure et demie environ) les élèves débattent, toujours en groupe, successivement de chaque affiche. L'enseignant intervient ici comme "président de séance". Cette partie est pour lui la plus délicate à gérer et fait donc l'objet d'une organisation très précise. Cette deuxième partie se termine par une "institutionnalisation" : à partir de ce qui est apparu au cours du débat, l'enseignant indique aux élèves ce qui doit être retenu. Dans ce cas, il s'agit essentiellement de l'insuffisance de l'appel au dessin pour conclure

Décrivons brièvement quels comportements nous nous attendons à observer chez les élèves et les hypothèses que nous désirons contrôler ou infirmer. Conformément à ce que nous avons observé lors de préexpérimentations, nous nous attendons à ce que les élèves commencent dès la phase initiale de travail individuel à tenter de construire le triangle à l'aide de la règle et du compas. Autrement dit, initialement, leur représentation du problème est la même que si on leur avait proposé l'exercice qu'ils connaissent bien : *construis un triangle dont les côtés ont pour longueurs 5, 9, 4.* Nous faisons même l'hypothèse que pour eux tout triangle dont on demande la construction existe : ceci est bien ce qui résulte de la coutume de la classe. C'est pour ne pas renforcer encore ce qui peut apparaître pour nous dans cette situation comme un obstacle d'origine didactique que nous avons choisi de demander si le triangle existe et non pas "peut-on construire" ou "peut-on tracer". Ainsi a priori nous pensons que l'idée que le triangle peut ne pas exister ne pourra résulter que d'une impossibilité rencontrée dans le dessin par certains élèves mais que la réponse majoritaire devrait être : il existe car nous avons réussi à le construire.

Notons que ceci implique un "blocage" dans certains groupes où l'unanimité se fera sur l'existence du triangle. Dans ces groupes, aucun débat de validation ne devrait apparaître puisqu'il y a accord et que l'hypothèse de non-existence du triangle n'a aucune raison être envisagée : le contrat didactique (c'est-à-dire les habitudes de fonctionnement de la classe) et la construction géométrique militent pour l'existence...Dans d'autres groupes au contraire, les positions devraient diverger et le débat s'engager immédiatement. Ainsi, les groupes devraient arriver dans un état hétérogène au moment de l'affichage puis du débat : certains déjà engagés dans le débat et ayant en somme la représentation du problème que souhaitent les expérimentateurs, d'autres pour qui l'affichage va changer la représentation du problème et qui devront s'investir dans le débat sans avoir éprouvé déjà leurs arguments au sein du groupe. Notons que ceci exprime certaines particularités de la situation de classe par rapport à une situation "de laboratoire" où les élèves auraient à chercher seuls ou en petits groupes : a priori on peut tabler dans une classe sur un éventail plus large de réponses.

Pour nous résumer, nous attendons de l'expérimentation d'une part l'émergence de certaines conceptions, essentiellement l'idée suivant laquelle, en géométrie l'appel au dessin doit suffire pour prouver une conjecture, conception que nous pourrions désigner comme une conception "réaliste" des objets de la géométrie. Nous espérons aussi que cette conception sera contestée par une autre conception plus "intellectuelle" qui arrivera à concevoir que dans ce cas les points doivent être alignés, mais si cette conception existe, nous espérons aussi découvrir par l'expérience sur quoi elle est fondée. D'autre part, nous avons conçu la situation pour qu'elle fasse apparaître des conflits socio-cognitifs entre les différentes conceptions et nous désirons vérifier l'existence de ces conflits et surtout leur effet.

4) Résultats de l'expérimentation

4.1) Les conceptions

On voit apparaître effectivement les deux conceptions attendues, sous des formes que nous allons préciser ci-dessous mais on voit en outre intervenir une autre conception, inattendue celle-là, suivant laquelle l'existence d'un triangle dépend de l'ordre de ses côtés. Revenons maintenant plus en détail sur ces trois points :

-la conception réaliste de la figure : elle se manifeste d'abord par la confiance dans le dessin : les premières constatations sont définitives pour ces élèves. Elle se manifeste ensuite par le désarroi quand plusieurs dessins faits par le même élève ou par des camarades se révèlent contradictoires : nous avons souvent observé dans ce cas une activité fébrile de construction d'un grand nombre de figures. Elles se manifeste aussi chez certains élèves par une position analogue à celle de Protagoras : deux cercles tangents (ceux que l'on trace pour construire le triangle) sont confondus sur une certaine longueur, ce qui permet d'ailleurs de régler éventuellement le cas de dessins contradictoires. Cette conception réaliste apparait comme très stable et résistante aux arguments avancés par les tenants de la conception intellectuelle : le constat sur le dessin a un statut très fort. Au niveau du langage, lors des débats avec les tenants de la conception intellectuelle les élèves réalistes feront appel au témoignage de la vue : "regarde" et à l'amélioration technique du dessin : "refais le, taille ton crayon". Le mot précis évoquera pour eux le soin apporté à la construction.

-la conception intellectuelle de la figure : nous caractériserons cette conception par la conduite suivante chez l'élève : très peu de dessins, voire pas du tout, ou des dessins symboliques qui ne proviennent pas de l'activité de construction du triangle à la règle et au compas mais en donnent directement le résultat anticipé par l'élève, c'est-à-dire un segment avec un point marqué. Au niveau du langage, lors des débats avec les tenants de la conception réaliste, ces élèves argumenteront en faisant appel à la réflexion : "réfléchis, raisonne" et à une figure idéale normale : "normalement, ça devrait se couper sur le segment". Le mot "précis" pourra se rapporter chez eux à un "dessin parfait", à la limite le dessin précis s'oppose à tous les dessins concrets. Lors des débats avec les autres élèves, la source de la conviction de ces élèves, lorsque nous avons pu la repérer est toujours liée à la perception graphique des propriétés du cercle : en langage mathématique, tout se passe comme si ces élève voyaient clairement les propriétés du type suivant : deux cercles tangents extérieurement sont de part et d'autre de leur tangente commune ou encore : l'hypoténuse d'un triangle rectangle est plus longue que chacun des côtés. Autrement dit, l'incertitude engendrée par le dessin sur le triangle proposé est levée en abandonnant le résultat concret et contradictoire de la construction et en se ramenant à une propriété dont on est sûr. Cette dernière certitude est elle même d'origine graphique (il ne peut en être autrement vu les connaissances mathématiques à cet âge) et l'attitude de ces élèves ne peut être qualifiée d'intellectuelle que parce qu'elle manifeste l'aptitude à lever l'indécision sur un dessin en le ramenant à une construction dont le résultat est suffisamment certain pour qu'on renonce à la faire. Ces observations n'ont malheureusement pu être faites que dans un nombre limité de groupes (trois sur huit) elle montrent en outre que chez les élèves réalistes, la perception de la figure peut être fort différente, en particulier les propriétés ci-dessus décrites ne leur paraissent pas du tout évidentes.

-la conception sur l'ordre des côtés se manifeste pendant le travail de recherche dans les groupes par la volonté de tester la construction en commençant successivement par les trois côtés proposés et en ne s'étonnant pas de trouver des résultats différents. La solution finale proposée par certains groupes d'élèves explique alors que le triangle existe ou non suivant le côté par lequel on commence pour le construire. Cette conception permet aux élèves de rendre compte de résultats de construction éventuellement contradictoires. Les attitudes des élèves sur ce problème sont très tranchées : pour certains il est évident que l'existence du triangle dépend de l'ordre des côtés, pour d'autres, c'est le contraire qui est évident. Nous avons pu constater que cette conception persiste chez des élèves de treize ans, dans la classe suivante, et une interview d'élèves nous a confirmé sa solidité. La recherche ne nous permet pas pour l'instant de donner des résultats sur ses relations avec les deux autres conceptions. Notons que, du point de vue mathématique, c'est un autre aspect du concept de figure qui est ici en jeu : une propriété d'une figure est en fait, en général, celle d'une classe d'équivalence pour l'action d'un groupe (ici le groupe des isométries).

4.2) Les conflits socio-cognitifs et leur évolution.

Ces conflits peuvent être observés à deux niveaux : pendant la phase de recherche du problème, dans les groupes où des conceptions opposées apparaissent, et pendant la phase du débat entre les groupes. Disons tout de suite que dans toutes les classes observées, les trois conceptions ci-dessus apparaissent au niveau des groupes, avec des dosages divers. Au niveau des affiches produites, la troisième conception n'apparaît pas dans une des classes observées.

Le conflit essentiel, qui nous intéressait de plus au premier chef, se situe entre la conception sensible et la conception intellectuelle de la figure. De manière assez surprenante on constate que, au niveau des groupes, toutes les issues possibles peuvent être observées : passage du niveau sensible au niveau intellectuel, statu quo sur chacune des deux positions et même, dans un cas, régression du niveau intellectuel au niveau sensible. Au niveau du débat dans la classe, c'est toujours la conception intellectuelle qui finit par l'emporter, même si elle était minoritaire au départ, mais il arrive que cette conception, même une fois institutionnalisée par l'enseignant reste seulement majoritaire, et non unanime : la conception sensible est bien disqualifiée chez les élèves, à la fois sur le plan cognitif, en ce sens qu'elle échoue à résoudre le problème posé et sur le plan social, par l'autorité de l'enseignant, mais chez certains élèves elle est simplement disqualifiée, mais non remplacée par une autre à laquelle ils n'ont pas accès. Dans ce cas, il y a en somme régression. Rappelons toutefois que cette étude ne porte que sur les phénomènes qui interviennent pendant les deux heures que dure l'observation et ne préjuge pas des évolutions à long terme. Soulignons aussi que, bien que la finesse des observations nécessaires exclue une étude de type statistique, ce dernier cas ne semble concerner qu'une minorité d'élèves. Il pose toutefois le problème de l'efficacité des débats cognitifs dans le processus d'apprentissage sur ce contenu.

La conception sur l'ordre des côtés pose des problèmes plus difficiles à résoudre. Elle n'entre pas à proprement parler en conflit avec les deux autres : nous n'avons pas observé de débat très approfondi dans les groupes à ce sujet, parce que les groupes se montraient souvent unanimes sur ce point, ou bien que le débat entre les deux autres conceptions y apparaissait comme prépondérant.

5) Conclusions

L'expérimentation confirme donc la présence des deux conceptions attendues mais en fait apparaître une troisième sur laquelle la recherche doit se poursuivre. Elle confirme en partie les résultats de N. Balacheff (1987) en ce sens que l'argumentation des élèves relève de "l'expérience mentale" lorsqu'ils partent de la conception intellectuelle et de "l'empirisme naïf" dans l'autre cas. Notons toutefois que les nécessités du débat amènent les tenants de la conception intellectuelle à argumenter éventuellement sur le terrain de leurs interlocuteurs, c'est-à-dire l'appel au dessin et que ici la procédure de raisonnement est imbriquée avec la construction du concept de figure et en est indissociable. Ces deux dernières remarques peuvent indiquer des causes de la limitation qui apparaît dans l'évolution des conceptions à partir des conflits socio-cognitifs. Une autre cause dont nous n'avons pas ici parlé en détail est la prépondérance fréquente des considérations de nature sociale (types de relations entre élèves) dans l'apparition des conflits et dans leur résolution. Enfin les résultats apparus à propos de l'étude de ce problème particulier et dont certains sont encore à confirmer, seront à confronter avec le comportement des élèves devant d'autres énoncés.

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THE ROLE OF CONCEPTUAL MODELS IN THE ACTIVITY OF PROBLEM SOLVING

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SUMMARY. The notion of conceptual model is used to study the activity of verbal problem solution in pupils (7-13 y.o.). Problem solving is described dynamically as a process of transformations from some basic intuitive models, integrated with pupils' culture, and expressed by natural language, into more elaborated and formal ones. The main features of such transformations are studied and used to get hierarchies for arithmetical and logical problems.

This report sketches author's research on conceptual models used by pupils (7-13 y.o.) in their activity of problem solving.

Following partially Lesh [81], a conceptual model is conceived as a system which integrates the organized mathematical knowledge (both concepts and processes) with the activity of problem solving in concrete problematic situations.

Empirical results and theoretical analysis show that problem solving is affected both by the deep mathematical structure of problems and by their linguistic context (namely, contextual, relational and lexical data). In a first time, pupils use basic and intuitive conceptual models, integrated with their culture; afterwards, they adjust, transform and translate such basic models into more mathematical and formal ones.

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A major point in studying problem solving is to investigate the transformations from basic to elaborated conceptual models. In the author's opinion, things are clear enough for arithmetical problems, but not so for other fields. Yet, some of these, e.g. logical problems, seem to be particularly stimulating and promising for clarifying the connections between (some) basic conceptual models and (the corresponding) more elaborated ones.

The report sketches firstly what a conceptual model is , showing how it works in the relatively well known field of arithmetical problems. Some significant examples are illustrated: namely, when transformations and translations of models are performed in conformity, neutrality or opposition w.r.t. the text and context of the problem; or when, in more complex cases, the phenomenon of hidden interference appears. It is so possible to define some examples of cognitive hierarchies for arithmetical problems, which are more dynamic and less involved than usual ones. The last part of the paper describes author's ongoing studies in the field of logical problems.

The research (still in course) has been supported by Consiglio Nazionale delle Ricerche (Grant #87.00958.01). A more expanded report for arithmetical problems can be found in Arzarello [88.1], [88.2]. The work on logical problems is still in progress.

1. Hierarchies on additive and multiplicative problems.

In a verbal problem very often there is a sort of dialectic tension between the (con)text of the problem and the conceptual model(s) used by a pupil in his or her activity of problem solving. This is the main reason why pupils (also those of low abilities) make transformations and translations from their basic conceptual models to new ones. In a basic models a pupil can represent the text of a problem directly: such a model is very intuitive and makes it easy to translate the text into procedures. E.g., in (subtractive) problems of separation with unknown result (a - b = ?), the pupil, particularly when of low mathematical abilities, makes use of a model of real (mental or physical) separation, etc..

1.1 Additive problems.

Looking more closely to additive problems, some of their intuitive models supply strategies which can be done also mentally (at most using fingers) for small numbers (it is the case of problems COMBINE, CHANGE), others do not (COMPARE, EQUALIZE; for the terminology, see Carpenter [85]).

In these last cases, for adequating their models to the new problem, pupils are inclined either to transform those at their disposal, or to translate them into more formal ones (1).

Basic models, which may or may not be stable, are essentially of two types:

(a) *Direct model*: e.g., in case a+b=c, a, b are the cardinality of some sets A, B, which are represented in some way, and then c is counted directly;

(b) *Counting*: in the above ex., one starts from a and counts +1 b times; the number so got is c.

The hierarchy results according to the complexity of translations and transformations needed to go from basic models to those effectively used.

A first group of transformations is made in *conformity* with the (standard) text of corresponding problems. A second group is not helped so much by the text: there is *neutrality*. Last, but not least, there are changes which must be made in *opposition* w.r.t. the text.

1.2 Multiplicative problems.

The basis of this hierarchy is in the analysis of Vergnaud (see also Nesher [87]).

Only main differences w.r.t. additive hierarchy will be pointed out:

(i) Pupils use formalized models more than in additive case; moreover there are less basic models (1).

(ii) To manage multiplicative problems, a more complex elaboration of the text is needed. This is the cause of two tipical facts, which seem to be missing in the additive case, that is:

a) the phenomenon of what I call hidden interference with the conceptual model used in the process of solution;

b) a closer connection between deep and surface structure of the text of a problem.

Problem solving activity consists essentially in reorganizing and adjusting each one's models to new situations. In the research, the technique of blocking a strategy, formerly available to pupils, or that of yeldying different strategies equally available, has been used. Such a technique is a main point in the activity of manipulating conceptual models and allows to read the processes of pupils in depth, especially at a formal level and to get more information on the type of models they use to solve problems. In fact, a direct analysis of their activities in problem solving is not possible when multiplicative problems are involved. So to say, in this case things are more dull than in additive one (where natural language is a good key to explain almost all processes of pupils); a more sophisticated analysis is needed, which takes into account also the logical and linguistic structure of the text and not only the mathematical one.

In fact, as the analysis of Nesher [87] shows, a lot multiplicative problem are based on a relation with two lexical arguments, and two numerical ones, let us say $R[A,B; a,b]$. Simplifying a bit, the standard structure of a multiplicative problem is:

$$R[A,B;a,b]$$

$$R[A,B;c,?].$$

However, in concrete cases, the logical and linguistic relationships may change: for ex., breaking the lexical and relational simmetry.

I call canonical the problems in standard form, non canonical those where logical and linguistic features have changed.

Pupils are very sensitive to such changes in the logical and linguistic structure of a problem. On the contrary, they generally show a good control of surface variations in the text (which reveals a remarkable difference w.r.t. additive problems). (Of course, they are as much sensitive to numerical difficulties, see for examples Bell et al. [84]).

Simplifying as much as possible, the main difference is the following. In problems where two different models are available (e.g., a partition division + a multiplication vs/ a quotition division + a multiplication), the percentage of choices is fifty-fifty, provided the problem is canonical, whilst in non canonical problems the percentage of models of partition vs quotition is 70% vs 30% (this happens even if calculations with the former are more difficult than with the latter). So in a case where metacognitive control is less good (as in non-canonical problems) pupils like better more stable models, as those of partition (or, in the analysis of Vergnaud, where a non-dimensional operator is used). This is a typical case of hidden interference: the different stabilty of a model is shown only when problems are more difficult. Models of partition (adimensional operators) are in fact more stable compared with quotition ones (dimensional operators); but this appears only in non canonical problems, so to say, at the second order. (Other forms of hidden interference are discussed in my quoted papers).

2. Conceptual models for some types of logical problems.

This paragraph contains the first results of an ongoing research, involving about 100 pupils from 9 to 13 years, and investigating the structure of the conceptual models that they use while solving certain classes of logical problems.

The term *logical* for a problem is here intended in a very wide range of the word, but discussion will be focused only upon problems whose solution requires the (concrete) use of propositional connectives and/or quantifiers laws in an essential way.

Examples: 1.In the island of knights and knaves (knights tell always the truth, knaves tell only lies) one of two people says :"Either I am a knave or he is a knight". Who are the two people?"; (see Smullyan [78], [82]).

2."Given six people, at least three of them know each other or there are at least three who do not know each other at all". (Typical example of a combinatorial problem which needs some logic to be solved).

Empirical tests with pupils and conceptual analysis show that (at least) two main families of models are necessary to attack such problems and moreover they must be integrated into a more flexible and general one:

a) Combinatorial models derived by the so called EXF schema for combinatorics, such as it has been analyzed and described in Vergnaud and Cohen [69], Maury and Rogalsky [70], Maury and Fayol [86]. The research is concerned mainly with models used by pupils for enumerating cartesian products in elementary combinatorial problems (product of measures, in the terminology of Vergnaud).

b) What I have called the truth and false model. It is not a basic model and further analysis is needed to investigate its more elementary parts. At the moment, it seems to split into two main branches, one related to combinatorial aspects of logical connectives (but is also deeply influenced by natural language), the other concerned mainly with the control on the truth/falsity of who tells what (see below).

Three major points will be sketched.

1. It is interesting to compare the use of models in purely combinatorial (verbal) problems with that in corresponding logical ones. In the first case, pupils generally use only combinatorial models, with many cases of gaps and failings, typical of such problems (e.g., see Maury and Fayol [86]). In the second, on the contrary, the two families of models are

closely integrated each other; the "truth-false" model is generally used as a control system on the whole (involving also metacognitive aspects). The conceptual content is so higher but, nevertheless, the two models are used more flexibly. To paraphrase Chevallard, the use of models with a higher mathematical content can make things easier. A typical example is given by strategies of solutions for problem 1. Generally, pupils argue by cases, using a model EXF(2,2) and use "truth-false"-model as a control system of their stream of resoning, in order to check out impossible cases. However, when the latter model is blocked in some way, uncertainty and failure increase massively: for example, in problems of type 1, if some people is "undecided" (that is data are not enough to find out who he really is), or in problem 2.

2. Performances of pupils in logical problems increase when they work in group and the teacher stimulates their interactions. When alone, they incline to use the two families of models very stiffly and find difficulties to pass from one to another. Empirical data show that this phenomenon is more pronounced for logical problems than for arithmetical ones.

3. The relationships of the model "truth and false" with natural language pose many problems. E.g., in problems of type 1, productive argumentations are based on model "truth-false", where an intuitive meaning of connectives is used; but when pupils are induced to use connectives as truth-table functions from the very beginning, most part of their argumentations is curtailed, and they use purely formal models with low control of the whole. Now, logical connectives and quantifiers are very often in opposition w.r.t. corresponding ones, as used and intended in natural language. This may be cause of hidden interferences in the processes of solutions of logical problems, even when pupils seem to be able to use logic at a formal level. Typical example is problem 2, which is a sort of "comedy of errors" for the way pupils (13 y.o.) understand negation of quantifiers (in domains with at most six elements!).

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HOW TO SEE EQUALITY IN THE DIFFERENCE: AN OLD PROBLEM WITH EDUCATIONAL VALUE

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Summary

Starting from two large-scale investigations of the abilities of children aged 6 and 11, we will look at the results concerning the ability to grasp similarities and regularities.

In the framework of curriculum development in primary school, the focus is on mathematical activities which unable the children to develop such ability.

Then the results obtained with two groups of children, whose teachers are developing such itineraries are provided.

Introduction

The purpose of this paper is an investigation of children's ability to discover hidden equalities in contexts where differences seem to be pre-eminent.

This investigation is mainly based on the critical analysis of tests, which have been administered to school children aged 6 to 11.

The educational value of this ability is generally known. It can be regarded both as a specific mathematical ability and as a general intellectual ability.

In fact, according to Wenzl (1934), a mathematical ability is the capacity to establish meaningful connections in mathematical material. Blackwell (1940) pointed out that mathematical abilities can be interpreted as abilities for selective thinking in the realm of quantitative relationships (quantitative thinking). Also Lee (1955) dealt with the ability to succeed in mathematics as the ability to understand the basic concepts of mathematics and to manipulate them.

On the other hand, the ability to see regularities is connected to the general factor "g" and a high correlation has been found between tests loading on the factor "g" and intelligence tests.

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The investigation we are carrying out originated from the analysis of a large-scale test's results. The data we got at the end of primary education revealed that children have a serious incapacity to grasp invariants. Consequently, a deeper investigation of children's abilities seemed useful at the beginning of primary school and a critical analysis of the mathematical curriculum in its entirety.

In particular, the Didactical Research Group (Nucleo di Ricerca Didattica) of the University of Pavia has been focusing its attention on the development in school of mathematical thinking and activity in its double face as an instrument to understand the real world and an opportunity to go towards abstraction. Didactic itineraries are proposed and evaluated.

I will give details as far as they concern the above mentioned ability to grasp equalities and invariants.

The origin of the question

Let me begin with just a few words to describe the ferment which is permeating Italian primary schools, in consequence of the arrival of the New Government Programs for primary schools. They were the result of the innovative routes drawn from pilot experiences and now represent a stimulus towards new directions.

In this framework, the necessity of knowing the real school situation, in which innovation should be inserted as suitable as possible was evident. In order to do this, the Institute for Research and Teacher Training of the region of Lombardia carried out an investigation of mathematical and linguistic abilities in school children at the end of primary school.

The test dealing with mathematical abilities (for the sake of clarity we call it M1) was equally administered at the beginning of the school year 1986/87 to a sample of 1500 pupils (aged 11) attending schools in Lombardia.

In consequence to the great variety of teaching styles, the test was conceived to be in accordance with a standard teaching model; that is a compromise between tradition and innovation.

It must be noted that the test, consisting of multiple choice items, did not suit the standard models of evaluation.

Let us focus on the following items, which are significant for our aims. They are reported together with their difficulty index, which is equal to np - 1 / n - 1 (where p is the ratio between correct answers and all answers given and n is the number of choices) if $n<5$ and is equal to p if $n\geq5$.

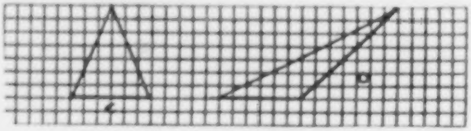
I Complete the following sequence

6 13 20 27 _ 41 _ _ D. I. 0.76

II Complete the following sequence

2 4 3 6 4 8 5 _ _ D. I. 0.35

III Look at the figures C and D



Only one sentence is correct. Mark the right sentence.

- the figures' perimeters are equal
- the perimeter of C is greater than D
- the areas of C and D are equal
- the area of C is greater than D
- the area of D is greater than C. D.I. 0.03

Children revealed a different behavior in facing items I and II. They mastered the sequence involving one additive operator (+7); this is probably due to the habit of doing similar things. Sequence II was not mastered by most children . They did not succeed in grasping the rule which was not immediately evident.

Item III was the most difficult of the test. Almost all children were not able to recognize the invariance of the areas in two very familiar figures. We suppose that the concept of area and the formula for its computation in the triangle were well known. The presentation of the item was not very familiar and this could be a disturbing element. However, the complete failure the children had in this item revealed their incapacity to apply their knowledge in a different context. The visual perception blocked the cognitive process. Children did not succeed in overcoming perception and seeing the hidden invariance. Hence, the stimulus to investigate if and how the mathematical ability to grasp similarities can be improved.

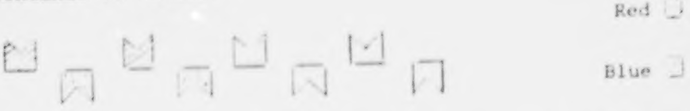
Let us consider the results of a similar investigation (called MO) that was carried out by our Research Group in order to get information about the mathematical abilities of children at their beginning of primary school.

This investigation was the starting point for research concerning the development of mathematical curriculum in primary schools. As is well known, at every level of instruction, the previous knowledge in one's possession has a basic role in organizing the new materials of knowledge. The New Government Programs for primary schools emphasize very much the importance of an initial investigation of children's abilities and recommend a systematic collection of data about the pupils' cognitive processes. The test MO was administered firstly to 300 6-year-old children at the beginning of the school year 1985/86. In its final version the test was administered to 480 children at the beginning of the school year 1986/87.

For the sake of clarity we call Group A the first group (N=300) and Group B the second group (N=480).

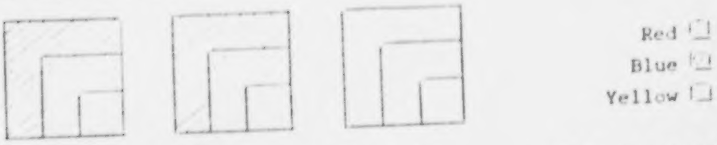
Neither group was randomly chosen. However we believe that they are, for our purposes, representative of the first grade children in the province of Pavia. The greater majority of our children had previously attended nursery school. The initial investigation (test MO) was conceived in accordance with familiar tasks that children usually face in kindergarten. The following items are significant for our aims. We also report the difficulty index for group A and group B.

1 Continue to colour



Gr.A D.I. 0.80 Gr.B D.I. 0.76

2 Continue to colour



Gr.A D.I. 0.82 Gr.B D.I.0.88

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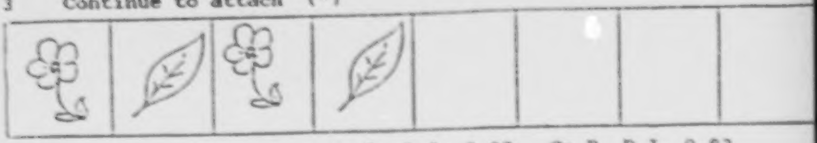
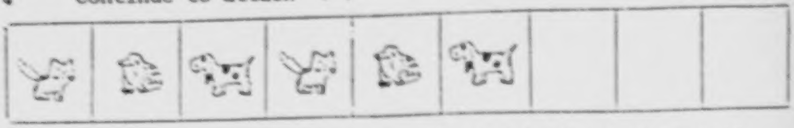
3 **Continue to attach** (°)



Gr.A D.I. 0.87 Gr.B D.I. 0.82

4 **Continue to attach** (°)



Gr.A D.I. 0.65 Gr.B D.I. 0.38

Children revealed a good mastering of rhythms. This ability was probably developed in nursery school. Nevertheless, the results are encouraging.
Another interesting item concerned the invariance of number. The teacher shows the child seven little sticks and asks him to count them. Then the teacher moves the sticks and asks the child: "How many sticks are there now?"
The answer is correct if the child does not count the sticks again. The difficulty index was 0.52 for group A and 0.04 for group B. We did not investigate the reasons underlying the difference we get. For our aims we considered the invariance of number to be a scarcely mastered ability.

The initial investigation M0, which I have only partially described, provided us useful information on the pre-school children's knowledge.
Then the proposal of a didactic curriculum and its experimentation in the classroom came as a consequence.
Without going into detail, we must note that it was highly recommended to emphasize activities devoted to the understanding of structural similarities in different contexts and situations. For example, teachers focused on sequences and rhythms, the use and analysis of the calendar and of temporal cycles, the introduction of arithmetic concepts by means of several approaches, the discovering of arithmetic laws.

(°) The children have at their disposition cards with designs.

Preliminary results

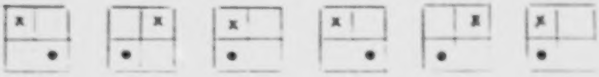
Our group faces the problem of evaluating the curriculum's validity in different ways. Basically, the evaluation lies

- in the results of objective tests, which are equally administered to all pupils periodically during the school year;
- in a qualitative analysis of children's products;
- in teachers' reports, on the ground of their experience.

We report here the results of objective tests: it must be noted that, due to the absence of some pupils, the two groups have not been tested in their entirety. However, we do not consider this relevant in the data collection.
The items have been established by all the researchers and teachers, who are taking part in the research, in accordance with the pattern of usual activities. For each item the percentage of correct answers seems to be a good measurement of the item's difficulty.

At the end of the second grade, the following item

Continue



was mastered by Group A at the rate of 84% and by Group B at the rate of 85%.
Thus, the results are convincing.
In the third grade teachers also gave emphasis to numerical sequences. At the end of the school year, the following item

Continue
0 1 3 6 10

was mastered at the rate of 65% (at the moment only the data of Group A are available).
At the beginning of the fourth grade, Group A faced the following items:

1) Continue

2 1 4 2 6 3 8 4

2) and explain the rule.

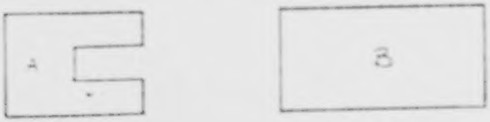
We got a correct answer percentage of 44.8% in item 1 and of 20.5% in item 2. These results indicate a rather good ability to see regularities. Sequence 1 was not easier than sequence II presented in the test M1 and it has been mastered at a higher rate by younger children .

We observe here that, as expected, the verbal explanation of the procedure is not an easy task. Just half of the children who mastered the sequence were able to describe the rule precisely. Some children gave perfect explanations: some of them saw the connection "number and its half", some others discovered the rule: -1, +3, -2, +4, -3, +5, ...

And, finally, an item which is located in the framework of geometrical activity . We administered it to Group A at the end of the first term of the fourth grade.

The item is:

Fred has two pieces of cardboard like these



According to you, are the perimeters of the two pieces equal or different? Why?

We had a correct response rate of 61%. Some pupils expressed very precisely the idea that the perimeters are equal even if they appear different.

Conclusions

Since the research is still in progress, it is obviously not possible to draw final conclusions. But we can make some preliminary remarks. When the pupils of Group A and Group B have finished their primary education, we will be able to give more precise results.

From the preliminary results has come the suggestion to insists on analogies and regularities every time possible.

We also think that this objective, which is present in most programs, is not always adequately translated into didactic practice. The attention to regularities is developed slowly but surely. This attention is the basis for overcoming many difficulties connected to the identification of structures, to the legitimization and use of symbols, and to the procedure of building an abstract model.

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EXPLORING CHILDREN'S PERCEPTIONS OF MULTIPLICATION THROUGH PICTORIAL REPRESENTATIONS

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A task based interview that focuses on children's recognition of multiplicative structure in pictorial representations is described. A rationale for its use and examples of children's performance are also given. Children's responses reveal differences in recognition of multiplicative structure from the perspective of several models: area, array, Cartesian product, grouping, number line, and stacks. Some children disregarded irrelevant data and/or modified the data to abstracted a multiplicative structure. Inappropriate responses included inattention to measure, a disregarded for meaning, a view of mathematics as written symbols, and attention to surface features.

The notion that children structure representations out of familiar elements is basic to constructivist perspectives on learning (Davis, 1984). These representations, as Vergnaud (1987) indicated, are shaped and refined by a whole range of experiences from initial, primitive concepts to advanced, complex ideas. If the conceptual development and refinement are based on a child's experience, then the range of those experiences is critical.

For multiplication, Anghileri and Johnson (1988) identified procedural rules and an equal grouping model as traditionally the most commonly represented in schools. While presently it is widely accepted among math educators that this model is too restrictive, student performance continues to reflect traditional practice. This is evident in related studies by Fischbein and his colleagues (Bell, Fischbein, & Greer, 1984; Fischbein, Deri, Nello, & Marino, 1985). They observed the inflexibility of the student model for multiplication and concluded that, for most children, multiplication remains linked to an implicit model, repeated addition.

Steffe and Cobb (1983) concluded that the iterative scheme is the essential component in the construction of the concept of multiplication (and division). They claim that the child structures experience in terms of his concepts, and that particular implementation of the multiplicative concepts consists of giving meaning to representation. If, as they suggest, concepts give meaning to representation, then exploring how children distinguish among representations could provide insight into their conceptual understanding.

The use of multiple embodiments allows children to demonstrate their understanding of a mathematical concept. However, it may be that certain models do not occur to the child. The range of variability and the breadth of understanding of a concept that do not occur are unlikely to be considered or explored. By their nature, pictorial representations add a prompt. Using pictures as *referents*, children have the opportunity to explain the external representation, or as Vergnaud suggests, *signified*, and then relate it by means of a *signifier*, their natural language, or any symbol they choose.

Certain questions emerge that suggest further study. Given a range of situations, visually different in terms of multiplicative representation, do children perceive any of them as related to multiplication? If iterative schemas are the essential material in the construction of multiplication concepts, can children recognize these in a variety of situations? In cases where they fail to see multiplicative structure, what distinctions do they make in terms of the feature of the picture? Can they recognize differences in representations that are not multiplicative models? And finally, how do they deal with these models, i.e., do they dismiss them or modify them in a way to impose a multiplicative structure?

In order to begin to explore answers to these questions, a task-based interview was designed and administered to children who are part of a city-wide mathematics project. The purpose of this paper is to describe the instrument, give samples of children's responses across the variety of multiplicative models considered, and suggest ways to classify children's behavior.

Background

Three clinical interviews were developed and administered by project staff to assess and monitor over time children's understanding of various aspects of multiplicative structure. From eleven K-8 elementary schools (8 public and 3 parochial), a representative sample of 60 children from grades 4-6 was selected and given two of three interviews designed to assess their understanding of multiplicative concepts. This paper describes a samples of children's performance on one of these, recognition of multiplicative structure from pictorial representations. Samples of children's performance selected from fifteen interviews are given.

Six models for multiplication (area, array, Cartesian product, grouping, number line, and stacks) were pictorially represented using ten related pictures (A-J) . Two other pictures (K-L) were included as distractors, suggesting additive situations. Figure 1 shows the pictures used.

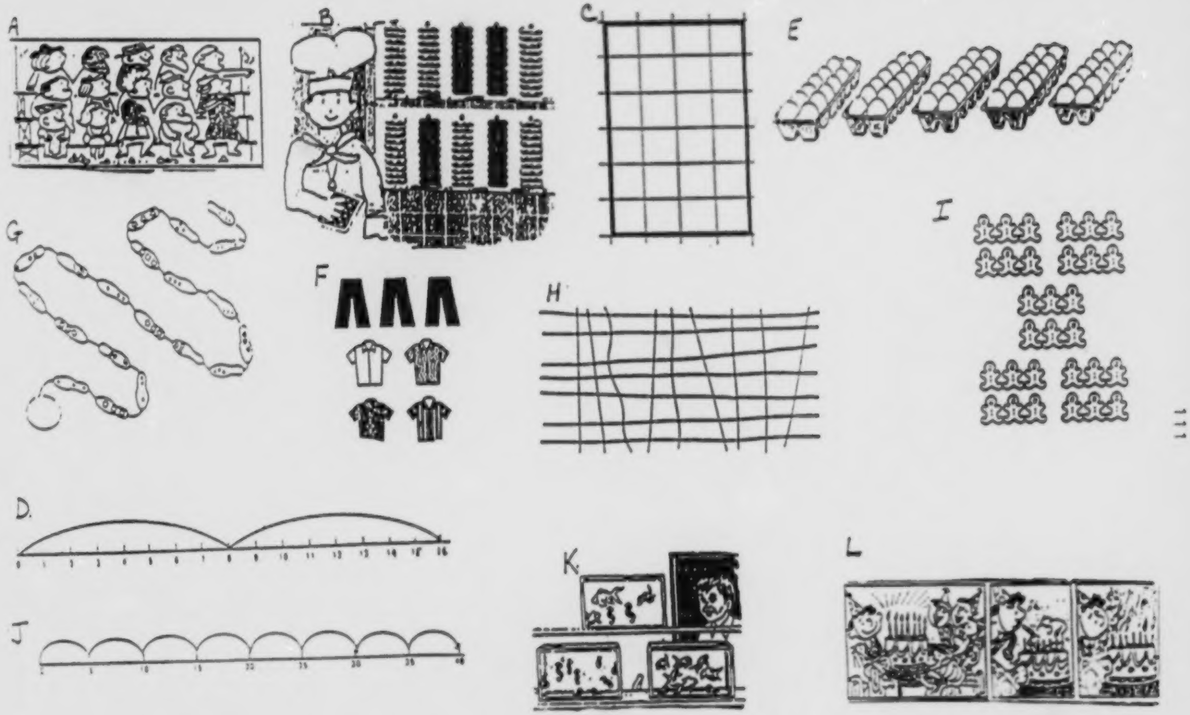
A
B
C
E
G
F
H
I
D.
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
J
0 5 10 15 20 25 30 35 40
K
L

Figure 1

A child was asked to pretend to be a teacher who wanted to explain multiplication to second graders. Then pictures were shown and the child was asked *which* could be used and, for the ones selected, *how* they could be used. For the ones rejected, each child was asked to explain *why*. Each interview was videotaped. Excerpts from these videotapes (some showing correct responses and others incorrect ones), clinician's notes, and children's written work provide the examples for this report.

Examples

Discrete Array (Picture A - the crowd)

Marita: **There is a baseball game..This boy wanted to know how many people in the stands, 3 times 5 equals 15...Let me see if I can group them up.. I'll group them up in sets of 5, there's 3** (encloses 5 people in 3 groups) **He knew 5 times 3 equals 15.**

Roberta: **...you could put the problem easy. Five times 3 equals 15 or 3 times 5 equals 15** (pointing to corresponding parts correctly)...**They could take this one** (Pic. A - crowd) **and think up a problem, and then show 5 times 3 on this** (Pic. J-5's number line). **Five - 1, 2, 3** (showing 3 hops) **3 times.**

John: **I would say: How many people in rows?...These 5** (pointing to top row) **times these 5** (pointing to 2nd row), **25.**

Al: **There's 3 people that are black and 12 that are white. Well, you can count them and see how many you have and then you get your total. Three times 12 equals 36 and you can count them 1, 2, 3, ...15.** Int.: What about 36? Al: **Well** (10 sec.) **It's just times tables. If you have it here, you'd go 12 times 3 equals 36 - there's three people in black. I don't know it. I can't get it** (counting) **15?** (colors hair) **I just colored in 5 people with blonde hair. There's 5 people with blonde hair; there's 9 people not blonde. Five times 9 equals 45 - but that's not the answer.**

Stacking and Grouping (Pictures B and I - donuts and gingerbread cookies)

Nicki (N): **It only shows adding. It doesn't show nothing about multiplication.** Interviewer (I): Show me where it shows adding. N: **Five her** (pointing to top shelf) **and 5 here** (pointing to bottom shelf) **and you just plus**

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them...unless it's 5 times 5 (indicating the same rows). **It equals out to 25. It's 5 plus 5 equals 10.**

Chris: **There are 10 groups and there are 3 people in each group. Each person (cookie) has 2 buttons. So now you have 10 groups,** 3 persons **in each group and each person has 2 buttons. Ten times 3 equals 30, so if there are 30 people and each person has 2 buttons it's 2 times 30 equals 60 buttons.**

Area: (Picture C)

Nicole: **Because it doesn't say nothing about multiplication. It doesn't give you no explanation on it and it just ...it's only blocks with squares and you don't know what its's telling you.**

James: **This picture of like some squares...this picture would come out to like a times table chart..you'd fill in numbers on it.**

Number Lines: (Pictures D and J)

Keith: (Referring to D) **Eight times 8 equals 16...no, 8 plus 8 equals 16 no, 8 times 2 equals 16.**

Keith: (Referring to J) **Can't use this for multiplication, but you can teach kids to count by 5's.**

Shawn: **All I know about these is adding and subtracting.**

Natural Groupings: (Pictures E and G - eggs and footprints)

Chris: (Referring to E) **There's always 12 eggs in a dozen...every time I go shopping with my mother, she always gets 12 eggs in these cartons. There are 12 eggs in each carton and there are 5 cartons. You multiply 12 times 5.**

James: (Referring to E) **There's eggs times the cartons. Five times** (counts every egg) **60. Five times 60 equals 300.** Int: Is 300 anything in the picture? James: (nods no).

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Lynn: (Referring to E) **Twelve** (indicating the eggs in the 1st carton) **times 4** (pointing to the last carton. Int: What would you say to a student who says, *Why did you only use these 4?* Lynn: **Because this one** (pointing to the 1st carton) **is the one that's going against all of them** (indicating the last 4 cartons).

Joe: (Referring to E) **I'd use this for - like 3 times 2..Like these 3** (pointing to the 1st 3 cartons) **times these 2** (pointing to the 4th and 5th carton), **6.** Int.: Where's the 6? Joe: **It's not there until you times the cartons.** Int.: Three times 2 will give you what? **Six cartons.**

Nicole: (Referring to G) **I rejected it...cause it's different. It has like adding. It just has pictures on it, like blocks with your shapes. It shows you shapes. Like it doesn't show no numbers, or nothing. And stars isn't part of math.**

Kim: (Referring to G) **It's adding, has plus signs in it.**

Distractors: (Pictures K and L)

Chris: (Referring to L) **You can use it for multiplication for candles, but you can't do it if you're going to tell how many are lighted** (pause) **because there are 3 cakes and 7 candles on each cake. Seven times 3 would be 21 candles.**

Robin: **Three times 7 equals 21. You could say 3 pictures were taken from a birthday party. All 3 pictures had cakes with 7 candles on them. How many candles in all? Three times 7 equals 21.**

Cartesian Products: (Pictures F and H - clothes and nonparallel lines)

Chris: (After identifying 12 combinations, the interviewer asked) Int.: So you have 12 different outfits? Chris: **No, but I could change it around 12 times. You change them like this** (rearranging them), **that's 3 times 4 equals 12.**

Observations

Children's solutions underscore the differences among them yet point to some commonalities. Some children seemed to see mathematics as a description of something sensible and others did not. Chris immediately recognized that there are 12 eggs in a dozen (making reference to shopping with his mother) and reasoned that 12 eggs in each carton and 5 cartons result in 12 times 5 eggs. He also clarified to the interviewer his meaning in the combination problem (that he did *not* have 12 different outfits but rather that he **could change it around 12 times.)** In contrast, John and Al seemed not to be concerned with what it meant to multiply *numbers of people by numbers of people*. Their inattention to measure and focus on abstracting symbols with which to operate led them to make up a multiplication problem that had no meaningful relationship to the picture.

Joe was not only inattentive to measure (he was comfortable in multiplying numbers of cartons by numbers of cartons), but seems to fall into a category described by R. B. Davis as *symbol pusher* (personal communication, January 26,1989). Notice Joe pointed to and multiplied the first 3 cartons by the next two to obtain 6; when he was asked where the 6 was, he replied: **It 's not there until you times the cartons** and indicated **six cartons**. Kim, for the footsteps problem, also appeared to focus on the symbols (she remarked, **it's adding, has plus signs in it.)** Lynn multiplied the number of eggs in one carton by the remaining number of cartons to produce her solution of 12 times 4. James recognized the quantity of 60 in the egg carton picture (he counted eggs) and multiplied the number of eggs by the number of cartons containing eggs (hence, his statement of 5 times 60 equals 300). Also, Nicki's attempt to relate multiplication to the donut picture resulted in her using numbers that could have referents in the picture.

An example of a child who successfully related the meaning in one model to another was Roberta. She recognized the appropriateness of *two* models, array and number line, and suggested how they might be related. Contrast this to Keith who recognized that Picture D could be used to teach multiplication but not Picture J, and Marita who changed the array model to one of equal groupings.

Some children were able to disregard irrelevant features and abstract a multiplicative relationship. Chris, for instance, suggested that if you disregard that the candles were lit, you could talk about 3 cakes and 7 candles on each and, therefore, use the picture to show 7 times 3 equals 21. Robin had a similar strategy and also disregarded lighted candles.

Other children (Nicki , Shawn, and Nicole, for example) did not recognize multiplicative structure. Nicole (in the footsteps picture) focused on the markings in the footsteps to conclude that **it doesn't show no numbers, or nothing.** Chris, on the other hand, seemed to recognize a three factor multiplicative relationship in the gingerbread example (recall that Chris said: **There are 10 groups and there are 3 people in each group. Each person has 2 buttons**) although he expressed this as a two factor relationship in two different statements (10 times 3 equals 30 and 2 times 30 equals 60).

Children attended to iteration differently. Successful students were not distracted by variations in visual elements, but those children who were unsuccessful seem to impose a requirement that a condition for iteration (e.g., size, color, group arrangements) was that the elements should be identical. Pictures provided situations which seemed to stimulate cognitive conflict and resulted in some children's inability to resolve differences between pictorial and symbolic representations. The variety of responses both within and across pictorial representations from this small sample suggests the benefits to be derived from a more detailed analysis of children's thinking in this domain. Work is in progress to extend the current preliminary analysis to a larger sample as well as to monitor the change in students over time and after an intervention has taken place.

Acknowledgements

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AFFECTIVE ASPECTS OF THE LEARNING OF MATHEMATICS IN A MULTICULTURAL SCHOOL*

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This paper addresses the question of what it feels like to learn mathematics at a multicultural school, Uthongathi, in KwaZulu/Natal, South Africa. 25 pupils from three race groups were interviewed weekly after school. All but four pupils (representing all three race groups) initially expressed positive feelings towards mathematics, but later interviews probed more deeply into feelings towards teachers, teaching methods, friends from other race groups, boarders and day scholars, and into pupils' perceptions of the relevance of mathematics in their daily lives and future careers.

Background: the school and pupils

The first of the multiracial New Era Schools Trust (NEST) schools in South Africa, Uthongathi, opened in January, 1987. Generous funding from the Chairman's Fund of Anglo American and de Beers ensures that pupils who can pass the entrance tests are not prevented by financial considerations from attending the school. A balance in numbers is maintained between pupils from various cultural groups, and between day scholars and boarders. The 25 pupils in the mathematics project, who were interviewed by the four researchers (Manjul Beharie, Yanum Naidoo, Anita Frank and project leader Norma Presmeg), were from Indian, Black and White race groups in standards 5, 6 and 7 (i.e., grades 7, 8 and 9). Thirteen pupils were boys and twelve were girls; fourteen were boarders and eleven were day scholars. Although, theoretically, socioeconomic status is not an excluding factor at Uthongathi, in practice the pupils tend to come from high or middle SES homes. Of the 22 participants who had fathers at home, eleven reported that their fathers had professional careers, and a further four had fathers who were managing directors in industry or commerce. Eleven of the mothers, too were professionals. So the attitudes and feelings expressed by these pupils may not be generalisable to pupils from different home backgrounds and they probably differ from those of pupils in segregated schools (as a follow-up study in 1989 will investigate). But one interesting aspect of the attitudes reported here is the extent to which race disappears over time, as a criterion for evaluating friends, classmates and teachers, in a school such as Uthongathi.

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* A paper by Presmeg and Frank reports on cognitive aspects of the Uthongathi research.

Manjul interviewed six above average achievers in mathematics in standards 6 and 7. She reported as follows:
"The favourite subject for most of them was maths. The average home that the participants came from was more or less middle class, with one or more parents being professionals either employed or self-employed (careers ranged from teacher or headmaster, to lecturers in science or geography at university, to marketing executives, lawyers, ministers, building inspectors and secretaries). These pupils seemed to be socialised into their own culture (Indian, African or Christian cultures) but were also exposed to a great deal of Western culture (technological culture)." Most of these pupils were envisaging training for professional careers, and their parents took an interest in their schooling. In interviews they were articulate, confident and self-aware.

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Several of Yanum's below-average achievers in stds 6 and 7 also considered professional careers. She wrote,
"It is interesting to note that all four males interviewed have ambitions that involve maths. Katide (std 7, Black) wants to be an engineering surveyor, Nilesh (std 6, Indian) a computer scientist; Sandile (std 6, Black) and Gragen (std 7, Indian) both want to become doctors. Lisa (std 7, White) wants to be a pilot and Cindy (std 6, White) wants to work with handicapped children. Lisa has moved from the traditional female career".
Unlike the other four pupils and the pupils in Manjul's group, two of these pupils, Sandile and Cindy, were rather quiet and withdrawn in interviews, speaking seldom or giving vague answers, and appearing to lack confidence. Both pupils were from fatherless homes.
Anita (six average achievers, stds 6 and 7) and Norma (seven std 5 pupils) also noted a tendency for their pupils to have professional career aspirations.

Attitudes to mathematics, teachers and teaching

Gragen is a confident and outspoken pupil, a boarder who feels "very comfortable" at Uthongathi. In his initial interview with Yanum he said, "I like maths, especially when you concentrate in class". However, in a subsequent interview in which pupils solved matchstick problems, the following conversation was recorded.

GRAGEN (While working with the first match problem, he hears another pupil speaking - easily disturbed): That's one good clever boy working with your friend.

I : So are you.

GRAGEN: No, I'm not clever. My father would like to think I'm good at maths; my father is a good mathematician, he

> got As from class 1, say from std 5 ... he did well. When I was in the government school I used to ask him to help with my maths homework, but then my tests, my reports, I did bad. ... I like to impress my parents.

Several of Yanum's pupils were ambivalent in their "liking" for mathematics. KATIDE said, "I just come to like it because people have been saying to me maths is the most important subject. If you don't do it you won't get a job - so I come to like it." LISA described the feeling of power which mastery of mathematics can give: "I like it when it's sort of basic. I feel sort of powerful when I can work out something. It's encouraging when you put a whole lot of numbers and it comes out correct."

Manjul's interviewees were more likely to speak of enjoyment of the challenge which mathematics provided:

TEBO (std 6, Black) : It's an easy subject. It like teases my brain, and I like maths exercises. It's a challenge.

KERRY (std 7, White) : I enjoy the challenge of calculating. Maths is necessary, a challenge.

With regard to her average-achievement pupils, Anita reported, "In the first interview, all six pupils insisted that they 'loved' or 'liked' maths, but as time went on, they started discussing problems they were experiencing in the subject." Sometimes these problems were associated with attitudes to the teacher or the teaching. NATASHA (std 6, Indian) said, "I don't understand some things. I try my best." Anita reported that Natasha depended heavily on rules in task-based interviews, as follows:

"This was evident in many interviews when she asked, 'What's

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that rule called?' In talking about maths she said that it's easy and logical because 'you just have to remember the rules'. Hence her difficulty in working out $\sqrt{9}$. When she has difficulty remembering rules, she takes a guess at the answer."

Natasha expressed very negative feelings towards her mathematics teacher. So, too, did TAMMY (std 6, White), with the same teacher:

TAMMY : We've got this maths teacher, Miss __. She doesn't explain properly and I don't know what's going on. She gets so cross with you, but I find them hard, this percentage. ... I've given up with Miss ___.

Anita reported, "Here again pupils of all races have difficulty with teachers, and in some cases with the same teacher. At Uthongathi, the pupils have all united against Miss ___ in an attempt to get rid of her. Although Miss ___ is White and so is Tammy, there is no racial bias on Tammy's part, she too spoke out against Miss ___."

By way of contrast, all four interviewers received favourable comments about Mr ___, an Indian mathematics teacher. The following comment is typical:

LISA (std 7, White) : I think he's brilliant. He uses as many methods as he knows because not all of us will be able to understand only one method. (Yanum's data.)

Both Daniel (std 7, White) and Tammy in Anita's group preferred visual teaching methods:

DANIEL : What I find helps is a teacher who uses the board a lot; if they draw on the board and instead of using just orals, they should draw. (Anita's data.)

Anita commented, "These teachers in multiracial schools especially should be aware of their personal cognitive styles and frames of emotional and value inferences, alter these if necessary, and use this cultural/racial and cognitive understanding constructively in relating to the individual needs of each pupil."

Attitudes to friends from other races, boarders, the school

All four researchers found that pupils named individuals from all cultures among their friends, but that there was a tendency for their best friend to belong to their own race group. BHAVNA (std 6, Indian) expressed the position articulately:

I : How do different race groups get on?

BHAVNA : When it comes to speaking and mixing we do that very well and there is no racial barrier there, but you would find in close friends we generally stick to the same race group. That's not done intentionally; in my opinion it's done because we come from similar backgrounds and environs and you enjoy certain things in life and you feel comfortable. (Manjul's data.)

Although some pupils commented that they had been lonely when first adjusting to the school, all expressed positive feelings about attending Uthongathi. The following comments are typical:

THAMI (std 7, Black) : It's good for us to study together, because it helps the other race groups to find out about other people. ... Maybe we think that they speak this other language and that they are different from me, so I shouldn't bother about them ---; they have different cultures ... but as you get to know them you see how they work out things and they see how you work out things and you can help each other. (Manjul's data.)

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MARC (std 7, Indian) : I have learned how to communicate; at first it was quite hard interacting with other race groups. It wasn't difficult but most people were wary ... like if you said something you were not sure how the other person will think/feel ... you may offend them ... but after a week we got used to it.

(Manjul's data.)

CRAIG (std 6, White) : ...It doesn't bother me at all now... I work with others just as if they were the same.

(Manjul's data.)

KERRY (std 7, White) commented on the "friendly, relaxed atmosphere" at Uthongathi. (Manjul's data.)

Yanum reported as follows:

"Uthongathi being the first multicultural school one would expect a lot of problems with pupils adjusting. However, this is not the case; the students are very mature and realistic on this issue. KATIDE (std 7, Black) had the following to say about being in a multicultural school, during the first interview: 'It's okay. I miss Soweto', and the following to say during the final interview (six months later): 'Last year I wasn't free ... now it's nice, I'm free, yes. I just feel as if I was at home. There's not much of a difference. It is not that much of a problem to me. At first I was not free and I was tense.'"

LISA (std 7, White) during the first interview said, "I like it ... . It's very casual. We were free to say whatever we thought here", and during her last interview with Yanum, when asked how she felt about mixing with other cultural groups, she said, " I don't actually notice it."

Pupils in Anita's group pointed out that rather than division into race groups at the school, there was a division into boarders and day scholars.

TAMMY (std 6, White, boarder) : There is quite a big division with some of them. I found that at first. This was before I had day pupil friends, now I don't find it very much. We're sort of like a bees' nest. We're inside the nest. The boarders are the bees, busy having their activities, and when the day pupils come, they're like stray wasps, and, you know, we get cross, and when they come in our dorm, it's our home and we don't like them imposing on us. It's like a dog's territory ... you feel it's yours.

The positive message of Uthongathi comes across clearly in the following extract from Yanum's data:

GRAGEN (std 7) has an Indian father and a Coloured mother and being at this school he feels very comfortable. "... I really enjoy mixing with other people ... no you see when I was small I really didn't worry about colour because my mom's a Coloured and my dad's an Indian ... so I don't really care about it. When I came to this school I feel more relaxed, I feel really happy ... something inside me. It's nice talking to all the race groups because you get all that racial feeling out of you. So I feel free ... communicating with all other racial groups... so you don't get this barrier sort of thing ... like you're White don't play with me. You know, once you get to know them they're much more friendlier than you think. It's just like to help each other and be one big family."

Acknowledgement. The researchers wish to thank all staff members and pupils at Uthongathi who made this research possible.

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A CONFLICT AND INVESTIGATION TEACHING METHOD AND AN INDIVIDUALISED LEARNING SCHEME - A COMPARATIVE EXPERIMENT ON THE TEACHING OF FRACTIONS

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ABSTRACT

In a comparative teaching experiment involving two classes taught fractions by the same teacher, a conflict and investigation method shows superior initial learning and good retention over two months, while a widely used scheme of individual guided-discovery booklets shows very poor retention

Introduction

The design of teaching has received attention in the plenary sessions of several recent PME meetings. In 1988, Nesher argued the importance of the teacher's role in designing learning environments - and for designs which ensure that common misconceptions are exposed and dealt with. Streefland showed how children's own creative productions in response to open challenges could involve them in much worthwhile mathematical activity. In 1985, Treffers and Goffree reviewed types of mathematics curriculum and argued for a realistic mathematics teaching rich in the transitions from reality to mathematical models (horizontal mathematisation) and in the development and exploitation of mathematical relationships and the processes of abstraction and generalisation (vertical mathematisation). At the same meeting Douady described a teaching sequence based on the 'tool-object dialectic', designed to provoke the construction of the notion of real number. In 1987, all the plenary lecturers were devoted to constructivism, and hence to a theory of learning, and by implication, of teaching. The research reports have been proportionately less concerned with teaching, but more with the description of pupils' conceptual development. None of the teaching theories discussed has been the subject of comparative evaluation against more traditional theories, though they were supported by internal evidence and argument, based on widely supported psychological principles. To encourage the spread of such ideas through the educational system it is important both to offer developed methods which a teacher could experiment with and also to present evidence to convince practitioners that the methods are worthy of the effort required to put them into practice. The present experiment aimed to compare the effectiveness of two teaching methods and hence to indicate the relative predictive power of two theories. Method A was a system of learning from a very popular scheme of individualised booklets, containing gently graded guided discovery material (SMP 11-16). In Method B, the pupils worked in groups of 3 or 4 at fairly hard challenges involving the production largely of their own examples; these challenges were devised by the teacher and the discussion of the groups' conclusions was handled by him.

Method A is related to the following theoretical principles: a) that time on task is a highly significant variable and is maximised by individualised work; b) that individual differences within a class in previous attainment and in learning capacity demand individualised learning tasks; c) that the experience of success is highly important in the development of mathematical competence and confidence. Method B depends on the principles a) that misconceptions need to be exposed and the conflicts resolved by discussion; b) that individual differences can be satisfactorily provided for by working with open-ended (and open-middle) problems and by pupils' generation of their own examples.

THE EXPERIMENT

In the experiment, two classes of 10-11 year old pupils were taught by the same teacher. The two groups were closely matched on the basis of the pre-test and on previous attainments. All had had similar mathematical experiences previously. This included some recent work on fractions.

The groups each had 9 lessons of about 50 minutes each over the course of three weeks on the topic of fractions. Group A was taught by using the individualised booklet scheme (SMP), group B by the method of investigation, conflict and discussion. The content of the booklets was taken as defining the field of work; the material for the conflict and investigation lessons was devised by the teacher, by choosing a set of critical problems capable of challenging the pupils, to cover the same [illegible]

TEACHING METHODS

Method A - individual booklets, guided discovery (SMP)

These booklets are attractively illustrated and refer to practical contexts wherever possible. In this unit, flags, farm animals, windows, cars and trains entering tunnels appear. The questions are gently graded and it is intended that pupils of all levels of ability should be able to work steadily through the material without meeting difficulties which retard their progress. A summary of the material appears in the list below. Each section of work consisted of some ten to twenty questions of the kind indicated, and in the case of numerical questions, often considerably more.

1. Meaning of fractions (plane regions)
2. Fractions of numbers e.g 1/3 of 12.... 3/5 of 15
3. Reverse problem: given part, find whole, e.g. picture showing 4 panes of a window, rest covered by a curtain;' visible part is 1.3 , how many panes in whole window?
4. Fractions of an hour. Minutes as 60ths.
5. Addition of fractions, 3/4 hour + 1/2 hour = ... ; 2 $^{3}/_{4}$ hour + 1 $^{3}/_{4}$ hours = ...
6. Comparison of fractions, e.g. 3/4 or 4/5? using strips
7. Equivalence (using strips)
8. Decimal equivalents, e.g. 4/10 is 0.4

Method B - Conflict/investigation method

The list below gives a summary of the material used and some of the additional questions bearing on the same point generated by the groups of pupils.

1. Meaning of fractions; need for equal part, e.g. Divide a square into halves in as many ways as possible. (Use squared paper). (Discuss, explain, justify).
2. Meaning of fractions, some equivalence, +, -, e.g. Share 3 bars of chocolate amongst 4 people .... 5 people. Share 2 $^3/_4$ bars among 4 people. Find as many solutions as possible. Make up own questions.
3. Comparison of fractions, e.g. Which is bigger, 4/5 or 3/4 ? (no method specified: pupils to discuss)
4. Equivalence, e.g. Find all fractions equal to 4/5; then groups make up own, e.g.

   Reduce: $\frac{16}{32}$ $\frac{56}{64}$.
5. Addition e.g. Find 1/2 + 1/3. (Methods used: Rectangles, Minutes, Equivalent fractions.)

Classroom organisation

In Method A, pupils worked individually at their own pace through the booklets; at the end of each booklet they marked their work from an answer book available in the classroom, and answered a further set of test questions. These were shown to the teacher, any difficulties discussed and then the pupils proceeded with the next booklet.

For Method B, pupils were arranged into four groups each of three or four pupils. Pupils were asked first to make their own individual written response to the given challenge, then to discuss this within the group and arrive at an agreed conclusion. Following this, there was a class discussion in which the conclusions of the various groups were put forward and the correctness and other merits of each solution considered.

OBSERVATION OF THE TEACHING

The teacher recorded the following notes about the progress of the work in Group A:

- Few difficulties, little or no stress, quiet and orderly working.
- A remarkable change in attitude and motivation from highly enthusiastic at the outset to bored and lethargic near to the end.
- Interaction between pupils less than usual, and little opportunity for class or group discussion, as the pupils were at different stages in their work. However, more teacher-pupil discussion than usual.
- Pupils of lower ability found security in having the rules and methods given to them by the booklets.
- Booklets encouraged an artificial enthusiasm to complete the work, more for the status of commenting 'I'm on Book 3, which are you on?" than for the satisfaction of learning and understanding.

In Group B:

- Teacher's role much more prominent, required an expertise in generating and guiding discussion, encouraging the shy to participate. Knowledge of the main misconceptions was also required.
- Pupils needed first to make a written response before any useful discussion could ensue.
- Pupils needed guidance in developing their ability to discuss.
- No boredom, but increasing interest and involvement.
- In the group contributions to class discussion, the spokesperson should not always be the most able or vocal.
- No limit on the level of difficulty. Often pupils were able to demonstrate their learning with quite difficult examples eg. cancelling 168/216
- Atmosphere sometimes noisy; stressful for teacher compared with Group A.

- Several pupils discussed the problems with parents.
- Most rated the work very interesting but quite hard.

THE TEST

The test was mainly taken from the test material provided with the booklet scheme; hence most of the questions were very similar to those quoted above. There were some items not explicitly covered in Method B. These were the reverse problem (given the part window, find the whole), the addition of mixed fractions and whole numbers, eg. $2^3/_4 + 1^1/_2$, the explicit connection of fractions with time (parts of an hour in minutes). The connection with decimals is also made in the booklet teaching but this does not appear on the test. On the other hand, the test contains a few items in which harder fractions appear. Some such fractions were treated in Method B, but not in Method A. Overall, the test had a slight bias in favour of Method A.

Comparing the content of the test with the content of the teaching of the two groups, we note that the test contains some material which is clearly covered in Method A but not, at least not explicitly, in Method B.

Attitude questionnaire

At the end of lessons 3, 6 and 9 for each group the pupils answered a very short set of attitude questions. These asked them to rate the lesson as 'very interesting', 'quite interesting', 'not very interesting' or 'boring' and in a similar way to rate how hard they had worked and how difficult they found the material. The general conclusions from these are given in the notes on observations of the teaching.

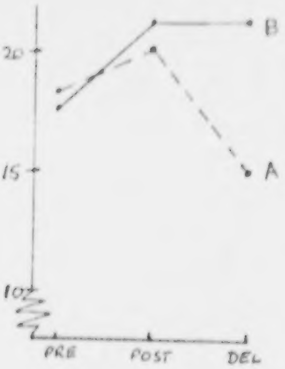
Test Results

For each group, the post-test was given immediately at the end of the teaching an the delayed test 7 weeks later, after the summer holiday. The results are shown in the table below;

RESULTS

	Mean Scores (Max 40)			Mean C[illegible]	
	Pre	Post	Delayed	Pre-post	[illegible]
Group A (N=14) Individual Booklets	18.3	20.1	15.0	+1.8*	[illegible]
Group B (N = 13) Conflict/Investigation	17.7	21.2	21.4	+3.5*	+3[illegible]

N is the number of pupils present for all three tests
* Difference on borderline of 5% significance
** Difference highly significant (at 0.1% level)



Discussion

When noting the dramatic fall in the performance of Group A to a level below that at which they began the experiment, it should be remembered that all of the pupils had had some fairly recent experience of fraction work before the experiment began; so both groups have probably begun at a level somewhat higher than might otherwise have been the case. However, the result shows clearly that the learning of Group B has been well retained, while that of Group A has been substantially lost. This is a result much more striking than than might have been expected but it is consistent with the observations made during the course of the teaching and it has survived our somewhat sceptical scrutiny. We believe this is a genuine result. We cannot claim it as a definitive and generalised result, since the experiment was performed with only one pair of classes. The matter is of considerable importance since the scheme from which Method A was taken is in use in a large proportion of British secondary schools. It has some visible merits in terms of presentation, and is generally well liked. However, a number of teachers have misgivings regarding the depth of learning which pupils achieve with it. There is clearly a case for more extensive experiments to be conducted, to see how far the results obtained here are typical.

But this experiment shows not only the comparative ineffectiveness of the individual booklets; it shows also the high level of learning, retention, involvement and enjoyment achieved with the method of conflict and investigation. This method deserves to be more widely adopted.

These are the practical outcomes. From the theoretical standpoint, the results call into question the theories on which Method A is based, that is that an appropriate way to respond to the wide differences in rates of learning and of prior knowledge which exist within a normal class is to provide individual materials. The conflict, discussion and investigation method provides for flexibility and creativity in the response of the pupils and thus generates an activity from which the class as a whole is able to learn more successfully. On the question of whether the exploration of a relatively small number of well chosen problems can promote learning which covers the whole field of the topic, the experiment shows that it can indeed do so.

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SOME RESULTS OF A LARGE SCALE EVALUATION OF THE NEW SYLLABUS AT FRENCH COLLEGE LEVEL

(6TH GRADE TO 9TH GRADE).

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Abstract:

The setting up of a new syllabus at French college level has provided the opportunity of undertaking a large scale evaluation of its implementation including the observation of the results achieved by the students. In this paper we will present only those observations which are more particularly linked with psychological questions. They essentially concern the difficulty of pinning down the notion of competency and the influence of certain variables on the results obtained by students: the general context in which the question is set and its relationship with other questions in the evaluation, the formulation used and also the stakes the evaluation has for pupils..., variables concerning the population taking the tests (age, gender, ...)

The aim of this paper is not to go as deep as possible on each subject (we reserve this for further papers), but only to provide the participants with an overview.

1) General description of the study.

Since 1986, in France, a new syllabus has been implemented at "collège" level (lower secondary level). It began with the 6th grade level (1986) then continued at 7th grade (1987) and at 8th grade (1988). The renovation of this syllabus will be completed this year with the implementation of the 9th grade syllabus.

Changes in the content are considerable and could be presented to those interested during the meeting. In this document we will only insist on the existence of an official list of "required competency" for each level. These lists, which, at each level, state more than 100 items of proficiency, will also be made avalable at the meeting.

From the very beginning, the French Association of Mathematics Teachers (8000 members), with the help of the Institute of Researsh on Mathematical Education of the University of Franche Comté has investigated with care the implementation of this new syllabus. In order to do this we made the decision to use as much as possible the human ressources of the Association, remaining in constant contact with the teachers involved, using many questions proposed by them, and returning the results to them as quickly as possible (results and first analyses within six months of testing). In fact the model used is close to that of responsive evaluation as presented by R. Stake.

2) The difficulty of grasping the reality of mathematical knowledge

Our main concern is to say something consistant about individual or collective knowledge. For a long time we have been becoming more and more aware that this is far from simple. Several previous studies (already published elsewhere) have helped us to highlight a set of difficulties in testing. Here are some of these points:

- The Place of a given item in a particular test can modify significantly the rates of correct answers. So can, for a given time attributed for the test, the general length of the test in which the question occurs.(by simple suppression or addition of other items).
- The operationnalisation (i.e. actual form taken by the question) chosen for checking a given skill is sensitive to such a degree that the change of only one word (even an article) can produce strong differences in the results. So, what can be said when question is "dressed" differently ?
- The formulation of questions may carry wrong ideas about the knowledge concerned. These errors can be due either to a lack of mastery on the part of the teacher-evaluator in the domain investigated or to his desire to make the questions simpler. In the two cases, wrong conceptions of students will not be detected and the validity of the operationnalisation used will not be assured.

- The formulation of questions may carry so much that is implicite that a correct answer is more a sign of similarity in the ways of thinking of the questionner and the answerer than an indication of knowledge.
- The information gathered depends, to a large extent, on the form of questionning. For a same a priori task analysis, according to the form a question takes: open, half open, shut or of Q.C.M type, the same students will not give the same answers. We have much evidence of the fact that the behaviour of a given student in one form of test can not be predicted on the basis of his results in another form. Only the a posteriori study of the procedures used by students can explain such a difference. Nethertheless the problem of determining what competency might be remains whole.

In the APMEP study, the tests used are built in such a way that the syllabus in its entirety is covered. Each student do not take all the tests (there are 10 tests and 48 different ways to participate to the assessment), but we managed to be able to linked most of the results.

As far as "requiered proficiency" is concerned, we observe that less than half of the students show mastery in more than half the questions asked. Fortunately, we took the precaution of linking our study with former ones, and, in all cases, the results observed now are better than those of the last ten years. So, it is not possible to incriminate a drop in students'level. At the very time when the authors of the syllabus think that they have reduced the requirements at these which should be attained by 80% of the students, we record that lesser than 05% of the corresponding questions meet this condition. Reducing the requirements to this of the questions really mastered would certainly have for first effect to reduce the teaching at a few meaningfullless tricks. So that is the actual possibility to enact official requirements which is put into questions.

Most of the tests are "puzzled", i.e, are made with questions taken in different areas, so that the mutual influence of items on each other is reduced. Nethertheless, in a few classes, "homogeneous" tests were used (i.e. regrouping questions from the same area). It is hardly surprising that in the case of homogeneous tests, the results obtained are much better, but this again leads us to reconsider the question of competency.

For an example, here is one question for which the difference observed is important.

When this question is included in a "puzzled" test, the rates of success is 29%. When included in a "homogeneous" test, the rates of success is 51%.

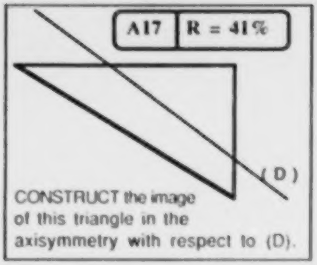
John buys a radio alarm clock costing 240F.
The shopkeeper gives him a reduction of 60F.

Express this reduction as a percentage

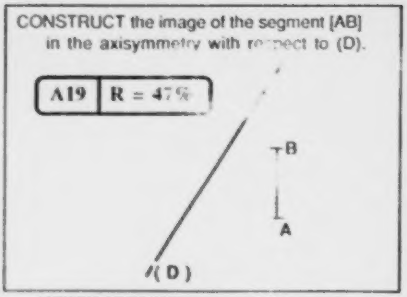
Answer :

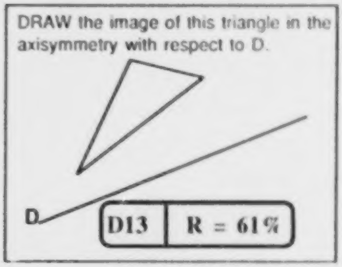
Item D17

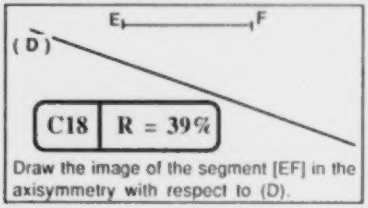
Hereunder, we show questions asked at 6th grade level, concerning axisymmetry. Contingency tables are presented when some students passed two of the questions (which is not always the case - for instance, no student passed both A19 and C18). The results confirm strongly what is already known (see Grenier,D.) But what about competency ?. When can we say that a student is *"able to draw the image of a segment line in an axisymmetry"*?



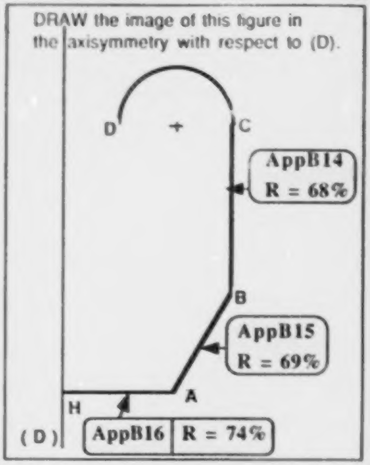
CONSTRUCT the image of this triangle in the axisymmetry with respect to (D).


CONSTRUCT the image of the segment [AB] in the axisymmetry with respect to (D).


DRAW the image of this triangle in the axisymmetry with respect to D.



Draw the image of the segment [EF] in the axisymmetry with respect to (D).


DRAW the image of this figure in the axisymmetry with respect to (D).

D
C
AppB14
R = 68%
B
AppB15
R = 69%
A
H
(D)
AppB16
R = 74%

In the contingency tables below, 0 indicates failure, 1 indicates success.

		A17	
		1	0
A19	1	30%	17%
	0	11%	42%

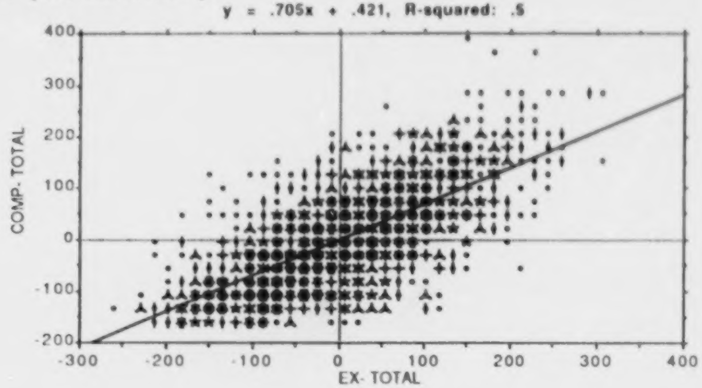
		A19	
		1	0
AppB14	1	36%	29%
	0	11%	24%

		C 18	
		1	0
AppB16	1	32%	43%
	0	09%	16%

What precedes is only an example. The study allows us to make such comparisons in many domains. Up to now, we have used a probabilistic implication coefficient between items introduced by Gras, R.(See Gras, R.). A study in progress is trying to give meaning to the notion of competency by using an extention of this coefficient to classes of items. The first applications of this method will probably be presented at the meeting.

Some of our tests called "required", aimed to cover closely the official list of required proficiency, while others, called "complementary" seek deeper skills. The obvious assumption that "who can do a lot can do a little" is more than often shown to be false.

Here is a scattergram of the results of a sample of 1600 students out of 1600 different group classes. It shows the relationship between their general results to required proficiency (EX-TOTAL) and to deeper skills (COMP-TOTAL). Indeed the correlation is high (r = .705) but we can estimate at 25% the proportion of students for whom the results obtained are quite contradictory.



3) The influence of student's involvment on the results observed

In our assessment, teachers had the right to use or not the students'results for their own evaluation . Our only requirement was that the students be warned beforehand.

One third of the teachers (out of the 1500 involved) seized this opportunity. The results show a significant difference between the two sub-populations. The study is not finished but we can hypothezise that, for a given question, the difference between the rates of correct answers of the two groups and the rates of non-answers is linked with the general difficulty of the question. We will have more complete results in July.

4) The influence of gender on the results observed

In our study, the difference between the results of girls and this of boys are highly significant, and this, in every area and for almost every particular question.

Questions C12 and C13 shown below will serve as a particularly sticking example.

The distance between Paris and Lille
is 300 km by motorway.
A truck takes 3 hours to cover this distance.
A car takes 2 h 30 min for the same trip.
What is, in km/h, the average speed of the truck ?
of the car ?

Item C12
Average speed of the truck :..............km/h
Average speed of the car :.................km/h
Item C13

Item C 12	**Boys**	**Girls**
Correct answers	73%	52%
Non-answer	20%	34%

Item C 13	**Boys**	**Girls**
Correct answers	22%	12%
Non-answer	30%	46%

It seems that the differences are particularly marked in the case of questions concerning everyday living: percentages, scales, areas, volumes... However this tendency is less marked if the questions are more abstract. We are studying the hypothesis that the form of the questions induces these differences. It would seem that as soon as the answer demands an explanation, the girs perform as well as the boys.

Why do we pay attention to this variable of gender? In fact, many other variables are observed, but this one takes on particular interest. First, it is unknown by ordinary class-room evaluation, secondly, while much more girls than boys enter 10th grade (49% out of a age class against 37%), much less girls than boys enter further mathematical studies (04.1% of a age class against 01.8%). In these conditions, any attemps to better understanding of the differences observed can be valuable.

In conclusion

As annonced, this paper gives some general information on a large scale evaluation , stressing on some particular points which will be more developped in the oral presentation and in further papers. Others points such as general context of teaching, assumptions made by the teachers, and so on... are also investigated, and in certain cases could be linked with the results overwiewed above.

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GRADE 8 STUDENTS' UNDERSTANDING OF STRUCTURAL PROPERTIES IN MATHEMATICS

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Abstract: It is suggested that students' difficulties in algebra are in part due to their lack of understanding of various structural notions in arithmetic. As part of a wider study, this report describes an investigation of Grade 8 students' understanding of commutativity and association for addition and subtraction, and their understanding of inverse operations. Findings to date indicate that these students have a very incomplete understanding of these ideas, which causes difficulty in distinguishing between allowable and unallowable transformations of numerical expressions. It is suggested that this may well hinder their understanding of algebraic simplification and manipulation.

The central aim of this study is to identify the extent to which students in Grade 8 have acquired understanding of certain structural properties in mathematics which may be considered fundamental to the learning of mathematics in general, and of algebra in particular.

Whilst ample evidence exists to show that students can have considerable difficulty in algebra with the meaning of letters and with various aspects of algebraic notation (e.g Küchemann, 1981; Kieran, 1981; Booth, 1984, 1988), difficulties have also been observed in students' understanding of various structural notions such as equivalence (Behr, Erlwanger & Nichols, 1980; Kieran, 1981; Filloy & Rojano, 1985; Booth, 1987) and the relation between operations and their inverses (Booth, 1987). Earlier work by the author has, in fact, highlighted the finding that students' difficulties in algebra are often due to pre-existing lacks in arithmetical understanding, rather than being related specifically to the use of letters (Booth, 1984, 1987). Such undersandings have been suggested to relate to the meaning of numerical (as well as algebraic) expressions, and to the relationship between different kinds of problems and the number sentences which model them (Booth, 1981, 1984; Brown, 1981)

Both these areas of difficulty can be viewed as difficulties in understanding various structural aspects of mathematics. In the case of the meaning of numerical expressions, the structural notions involved may include association, commutativity, distribution, and the relationship between operations. These properties are important both in understanding the meaning of the expression itself, and in distinguishing those transformations of the expression which are allowable from those which are not. In the case of modelling a problem situation by a mathematical expression, what is required is a recognition of the structural isomorphism between the problem situation and the mathematical statement appropriate to its solution. The relevance of both kinds of notion to successful performance in algebra is clear. Students who do not understanding the structure of numerical expressions and their allowable transformations may well not understand the same transformations applied to algebraic expressions. Similarly, students who cannot represent an arithmetical problem by an appropriate numerical model may be unlikely to understand the generalised (algebraic) model for the class of problems for which the particular arithmetical problem in question is an example.

Clearly, if the essence of algebra is attention to and understanding of the structural nature of mathematical relationships and procedures, then students who lack this appreciation are likely to have difficulty with the subject. The study reported here aimed to investigate this idea by examining students' understanding of certain structural features fundamental to algebraic thinking, and to explore the implications of the identified levels of understanding for students' performance on various algebraic tasks. This report is confined to one aspect of the study, and focuses on students' understanding of inverse operations, association and commutativity. Other aspects of the research will be reprted elsewhere.

METHOD

The general model followed was that of survey testing followed by individual interview. The purpose of the testing was to obtain information on the kinds of replies that students give to tasks requiring them to make specific use of the structural ideas under study (see Table 1), and on the prevalence of the various kinds of response. The findings from the testing were then elaborated and supplemented by data from individual interviews with a subsample of the students tested

The data reported here are for 120 students from four mixed-ability Grade 8 classes (age 13 years) The students have all commenced algebra within the context of an integrated curriculum in mathematics and are at the end of their first year of such study. The typical algebra content studied includes substitution, simple linear equations, and algebraic simplification and manipulation. All 120 students completed set A and items B(1) and (2) (see Table 1). Half the students then completed items B(3) to B(7) (n=60), while the remaining students completed set C (n=60). The tests were completed within one half-hour session.

Following administration and analysis of the class-tests, a subsample of 20 students were selected for interview on the basis of their answers to the test items.

RESULTS

Table 1 shows the percentage of specific categories of response to the test items under study.

FINDINGS

Inverse Operations: Less than half the students correctly selected an inverse operation without recourse to numerical substitution, even in the 'simple' cases of addition of 3 or multiplication by 6 (items A(1) and A(2)). Substituting a numerical value for the variable usually resulted in a correct response for the addition item, but not for the multiplication item where the common response was a subtraction, the magnitude of which depended upon the original substitution value chosen. In the items involving two different operations (items A(3) and A(4)), the proportion of correct responses was much lower (3%) Those students who had answered the single-operation items correctly but not the two-operation items typically gave one of two responses: one indicating a knowledge of inverse operat-

TABLE 1

A ITEM			RESPONSE						
			Correct	Substitutes for □	Impossible	Wrong order	Simplifies	Other	Omits
In this puzzle, I think of a number, , and then do something to it as shown. What must you write in the space if we want to get back to the same number I thought of, as answer? (The puzzle must work no matter what number I think of.)	A1	□+3 . . .=□	40	40	3			5	12
	A2	□x6. . . .=□	35	38	7			7	13
	A3	□x3-2 . .=□	3	38	7	22	11	7	12
	A4	(□-2)x3 . .=□	3	38	7	25	8	5	14

Do the next question WITHOUT CALCULATING. Draw a line linking the expression in column 1 to every expression in column 2 which you think is equivalent to it (there may be more than one equivalent expression). If you think no expression is equivalent, tick NONE. If you would need to calculate first, tick CALCULATE.

B	ITEM	RESPONSE						
B1	749+928	928+749	928-749	749-928	None			
		95*		2	3			
B3	583+247+98	247+583+98	583+98+247	98+583+247	98+247+583	All		
		10	10	7	13	60*		
B4	928+749-684	928-749+684	928+684-749	[illegible]		None	All	Omit
		7	17	30*		43		3
B5	19.7-3.8+7.4	19.7-7.4+3.8	19.7+3.8-7.4	3.8+7.4-19.7	7.4+19.7-3.8	None	Calc	Omit
		33	10	17	10*	17	7	6
B6	417+591-825	417-825+591	591+417-825	825-591+417	[illegible]	None	All	Omit
		7	43	13	3	10	3	14
		7*						
B7	15.6-8.9-3.4	15.6-3.4-8.9	8.9-15.6-3.4	3.4-8.9-15.6	Omit	None	Calc	Omit
		30*	17	3	3	23	10	14

	ITEM	RESPONSE						
B2	852-684	852+684	684+852	684-852	None	All		Omit
		3		25	63*	2		7
C3	583+(247+98)	(583+247)+98	(247+98)+583	583+(98+247)	All	None		Omit
		3	17	10	67*			3
C4	928+(749-684)	(749-684)+928	(928+749)-684	928+(684-749)	All	None		Omit
		13	3	13		4		
		7*						
C5	19.7-(3.8+7.4)	(19.7-3.8)+7.4	(3.8+7.4)-19.7	19.7-(7.4+3.8)				Omit
		10	60	33*				3
C6	417-(591+825)	(417-591)+825	(591+825)-417	417-(825+591)				Omit
		10	50	40*				7
C7	15.6-(8.9-3.4)	(15.6-8.9)-3.4	(8.9-3.4)-15.6	15.6-(3.4-8.9)	All	None	Calc	Omit
		30	40	23		13*	4	

* denotes correct response. For items A1-4, B1-2, n=120. For B3-7 and C3-7, n=60

Where percentages total more than 100, students have selected more than one answer ions but not of the order in which operations have to be undone (as in the response □x3-2÷3+2, see item A(3)), and one in which the numerical part of the expression was simplified (hence showing a lack of awareness of the meaning of the initial statement) and then the inverse of the resultant operation applied (as in the response □x3-2÷1, where the numerical part 'x3-2' had been first simplified to 'x1').

Commutativity and Association: Most students (n=120) seemed to be familiar with commutativity of addition (item B(1)). 25 percent also erroneously extended this to subtraction (item B(2)), supporting findings documented elsewhere (e.g. Booth, 1984). When more than one operation was involved, however, students appeared to show considerable confusion over the meaning of expressions, as evidenced by their apparent difficulty in distinguishing transformations which maintained equivalence from those which did not. For example, while the expression containing only addition was clearly easiest for the students completing the rest of set B (n=60), with all students selecting a correct alternative for 583+247+98 (item B(3)), only 60 percent appeared to recognise that all the given expressions were equivalent. The expression containing only subtractions (item B(7)) was less well handled, with only 30 percent selecting the correct alternative, and this proportion dropped to approximately 10 percent in the case of expressions involving both addition and subtraction (items B(4) to B(6)). Evidence from the interviews showed that students tended to focus on only part of an expression, ignoring its function within the expression as a whole. For example, 33 percent of students gave the response 19.7-7.4+3.8 as equivalent to 19.7-3.8+7.4 (item B(5)). Those interviewed typically achieved this by focussing on the part '3.8+7.4', which they were happy to commute to 7.4+3.8 regardless of the context in which it appeared. Students were, however, also able to consider more complex 'parts'. In the same item, for example, 17 percent chose 3.8+7.4-19.7 as an equivalent alternative, apparently by treating the original expression as 19.7-(3.8+7.4) and commuting this (erroneously) to (3.8+7.4)-19.7. Many of the same students similarly chose 684-928+749 as an alternative for 928+749-684 (item B(4)).

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Other findings of interest from the interviews included the belief by students that while addition expressions could be rearranged at least in one form or another, there were no rearrangements possible for subtraction expressions (see item B(7) where 23 percent stated that none of the alternatives was equivalent). For some students, this belief also extended to expressions containing mixed addition and subtraction operations (e.g. item B(5)). Also of interest was the finding concerning the need to calculate. Whilst conceding that it might be possible to distinguish between whole number expressions which were equivalent and those which were not simply on the basis of inspection, some 10 percent of the students considered that this could not be done in the case of expressions containing decimal fractions. In the latter case, it was considered necessary to calculate the values first, before equivalence could be determined. This perhaps provides further evidence that some students do not appreciate the strurtural similarity between expressions containing different kinds of elements.

The items containing brackets (set C) were generally handled more successfully than those without. This appeared to be due to the fact that the brackets allowed students to carry out their tendency to 'clump' the expressions into parts more easily, by actually providing the parts for them. Inspection of the correct alternative(s) in each item shows that in almost every case the integrity of the statement in brackets is maintained. The only exception to this is the alternative correct expression (28+749)-684 in item C(4) in which the brackets themselves are rearranged. The proportion of students selecting this as correct was a relatively low 10 percent. During the interviews, students typically rejected this option on the grounds that "you can't move the brackets".

In set C as elsewhere, the tendency to commute subtractions was maintained, with approximately 30 percent of students selecting a commuted subtraction in each item (items C(5) to C(7)). Interestingly, there was a much stronger tendency to commute the whole expression (e.g. (8.9-3.4)-15.6 for 15.6-(8.9-3.4), 40%), rather than commute subtractions within brackets (e.g. 15.6-(3.4-8.9), 23%). Explanations given in interview mainly related to the belief that while it was not allowable to commute single subtractions, this was allowable in the case of complex expressions involving subtraction. There was some suggestion that this in turn was prompted by a reluctance to allow an expression such as 3.4-8.9 (which students can 'see' as being impossible in terms of the belief that "you can't take a bigger number from a smaller one"), whereas it is not so immediately obvious to the students whether (8.9-3.4)-15.6 is of this nature or not.

Overall Performance: In terms of overall performance, only 3 out of the 120 students gave correct responses to all four of the items in set A; none of the subset of 60 students completing set B had every item correct; and similarly no student gave completely correct responses to all items in set C, although two students were correct but for the omission of one of the correct alternatives in item C(4).

CONCLUSION

The general finding from this investigation seems to be that many Grade 8 students have only a very incomplete understanding of the meaning and allowable transformations of numerical expressions involving addition and subtraction. This would not appear to stand them in very good stead for their introduction to algebraic manipulation, and indeed indicates a less than desirable level of numeracy as currently defined by many mathematics syllabuses. Whilst the importance of structural understandings of the kind addressed in this study has been clearly recognised in past syllabuses (witness the stress placed on 'number laws' in many of the now defunct 'modern mathematics' programmes), the problem was that educators found no successful ways of helping students to understand them or to appreciate their significance. Consequently, it was easier to remove the ideas from the syllabus as too formal, rather than try to battle with the challenge of making the ideas themselves more intelligible. In view of the ongoing problems which students have in algebra and in many other areas of formal mathematics, it may be that we made the wrong decision.

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UNCONDITIONAL MATHEMATICS

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Based on case study research with students studying to be elementary teachers, the authors propose a view of mathematics education that questions the assumptions surrounding mathematics and teaching. It is a view that centers around the process of gaining trust in self, in students, and in mathematics as a subject. Pedagogically such a view leads to students and teachers who can think creatively, have a higher mathematical self-esteem and are autonomous individuals.

1

Mathematics is based on conditions and assumptions, the questioning of which led to non-Euclidean geometries (Kline, 1982) for example. But to do such questioning, one needs to feel unrestrained or bound by the conventional limits, needs to believe that such limits were created by other humans, and therefore can be questioned and altered. It is this attitude of not feeling bound by conventional limits or assumptions that we view as "unconditional mathematics". It is a view that will be described here in terms of mathematics education, especially as it relates to students studying to be elementary teachers.[1]

The point to be made is that many student teachers (and teachers) feel bound by the educational conditions placed on them by what they see as experts --- the power and authority of textbooks or curriculum guides for example. But not only are they bound by such conditions, they have little trust in themselves as decision makers. This lack of trust intertwined with fear of mathematics gets manifested as a clinging to the textbook, curriculum guide, and lesson plan. This in turn leads to a mechanistic style of teaching, and hence learning.

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[1] This research began when the second author was a student teacher and began participating in a long-term case study with the first author. This research has been extended to include case studies of other student teachers as well.

We propose a view called "unconditional mathematics", a process requiring trust in self, the student, and mathematics as a subject. Such trust involves being able to take risks, to make mistakes from a position of courage not fear, to take a "leap of faith" (Kierkegaard, 1985) that a flexibly bounded teaching process will evolve into student learning. In the rest of this paper, this process is explicated, with the reader's realization that brevity is of necessity here.

2

Trust in self is a critical element of this process, because it refers to the teacher in the classroom. The process of teaching involves an awareness and a consciousness sensitive to what is occurring with the teacher and between teacher and students. This sensitivity could be labelled "living in the moment", a state that is marked by no conscious awareness of past or future, a state that is at the heart of the practice of Zen. (See for example, Hanh, 1975, 1987, 1988.) What is important about living in the moment is that one cannot possibly be fearful or anxious in the moment because such fears surface when we think about the past or future.[2]

In terms of teaching, a "living in the moment" state manifests itself as a flexibility, a flow of instruction that depends on observations and acknowledgments of what is occurring in the teacher, the students, and in the interaction between students and the teacher. It requires a lesson plan of course, but importantly requires a release, a letting go of such a plan to allow for spontaneity.

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[2] Living in the moment acknowledges that the past is part of the present. So to live in the moment one needs to integrate past and future. For example, memories may surface during the present and part of the moment is somehow acknowledging those memories.

For example when Anthony, a student teacher, taught mathematics, his fear and anxieties took over. He clung (literally) to the textbook, watched the clock constantly, worried about whether or not he was covering his lesson plan. He seemed totally unable to catch those student comments or questions that would lead him and the class astray from the "plan", and yet would have led to better teaching and learning. The sensitivity to student confusion and understanding that was so evident in other subjects disappeared when he taught mathematics. He attributed this to his uncomfortableness with the subject, as he said, "the last area of his life that needed breaking through".

How does one break through such fear?[3] We see such a process as rooted in Jungian psychotherapy and as first involving an awareness of such fears, then the acknowledgment of them to the self and to others by giving voice to them --- explicitly articulating the fears, and if possible in the context that evokes them, the mathematics classroom. For example, by honestly voicing his uncomfortableness with mathematics WITH the students he was teaching, Anthony would have been on the way to overcoming his fears.

To be able to do so however, Anthony would have needed a different way of thinking about himself as a teacher and about his students. Instead of viewing the teacher as the expert and authority, an oppressive view of "power over" students, a liberatory view rooted in dialogue between teacher

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[3] Of course there is a vast research literature dealing with the entire "math anxiety" movement. Suffice it to say that this is not the place to deal with this literature. What seems to be lacking however, are many detailed accounts of individuals who conquer such fear and descriptions of the process that allowed them to do so.

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and students is needed (Freire, 1970).

Such a liberatory view however runs contrary to traditional notions of power and authority, notions psychologically rooted in a spiritual tradition that is fall/redemption based, and hence teaches fear, rather than being creation-centered, and hence teaches trust. That is, fall/redemption based spiritualities are patriarchal, do not see faith as trust, emphasize control rather than letting go, emphasize obedience rather than creativity, and live in a concept of time as past or future rather than the now and making the future happen now. (See Fox, 1983.)

3

When a teacher is able to live in the here and now, there needs to be trust in the students. That is, living in the moment frees the teacher from lesson plans, curricula, the set norms of educational conditions. There is a trust that by planning and then letting go of the plans, that students will learn what is necessary for them to learn, that mistakes and failure are part of the learning process.

By exhibiting such trust, teachers allow students to make their own mistakes. They acknowledge and accept errors with the purpose of using them as learning devices, much as the process advocated by Mason, Burton, and Stacey (1985).

In an atmosphere of fear however, there is little allowance for error and for viewing errors as forms of learning. Fear leads to the need for one right answer, to rewarding the obtainment of this answer and the punishing of any deviations from it. The answer is often the one given by the external authority --- the textbook. Thus, there is little self-trust, with little building of an internal mathematical sense. The process of solving problems in *Thinking mathematically*, a book that often offers no solutions, has led many methods students to develop internal trust. At first the process is frustrating because they are used to being able to find answers "in the back of the book", but then they stop looking for them and begin a process of exploration of and awareness of their own knowledge.

4

We are led to trust in mathematics. This is especially important when assumptions and conditions are released and are held up for question. Here students are asked to think critically about what number is or about their definition of mathematical equality, for example. They must draw on their own experience and trust in their knowledge, not in an external expert. However students and teachers must also trust that mathematics is not chaotic (Gleick, 1983) but somehow orderly. When open-ended questions are posed, ones that question assumptions and/or ones that could have more than one right answer, students learn that there are not an infinite number of answers. But such a situation evokes great fear in studentss As one said, "the next thing you'll be telling us is that 2 + 2 = 5". She seemed to think that either mathematics means there is one right answer or there is chaos.

Such an attitude may again be related to what Fox (1983) speaks of as a fall/redemption spirituality verses a creation-centered one. That is, the former deals in dualistic thinking, either/or, whereas the latter deals with dialectical thinking, both/and. Many students have trouble thinking dialectically, envisioning that mathematics can have problems that have one right answer, at the same time have problems that have more than one right answer, and still not be in utter chaos.

What is the pedagogical need for an unconditional mathematics? We see at least three intertwined reasons here: creative thinking, mathematical self-esteem, and autonomy.

Creative thinking involves breaking boundaries, questioning assumptions, posing new theories. Advances made in science from Copernicus to Einstein to quantum theory (Hawking, 1988) have involved such questioning. The attitude that has allowed for such breakthroughs must be related to a high degree of trust in self as we have described here, one that would create a high level of self-esteem, a sense of autonomy.

In conclusion, Kathryn speaks to some of the issues in this paper, in relation to her student teaching experiences:

In examining my own process as an educator with a math anxious personality I have learned to accept where I am in relation to the subject area, that is, acknowledging that I may not always know the answer. This acknowledgement occurs within myself in the classroom and in relation with colleagues and peers. The risk involved in doing so both privately and publicly is continually rewarded by outside stimulus that keeps me consciously moving along in my math process.

As I focus on my process in the classroom I bring all of myself into the teaching experience. This is also a risk. It would be so easy to leave the anxious part behind and stand before my pupils and colleagues gathering strokes for being an expert, while at the same time using this expert role as a cloak behind which to hide my anxiety.

Questioning this expert role involves the realization that any given discipline has an inherent process which grows from cultural or social conventions. For example, we would all agree that 1 + 1 = 2 because we accept certain assumptions that make it so. If we did not operate from a collective agreement that expects the symbol 1 to operate in a certain way

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and trusts that 1 represents what we all agreed upon, we may not be so ready to agree that $1 + 1 = 2$. A student for example might argue that $1 + 1 = 11$. We, as teachers, might jump in to explain place value, coming from the premise that WE know the answer and therefore the student is wrong and must adhere to our assumptions to become a part of the collective. However, part of dissembling this expert role lies in trying to understand the student's logic, the student's assumptions, ones that led him/her to say that $1 + 1 = 11$.

In the classroom, the most important aspect of a process focus was experienced in the connection with my students. As I asked myself what mathematics means to me, I asked my grade 6 students what it meant to them. Because we were all asking the same question, the expanse of time and experience was bridged in precious moments of sharing what math means to us in our daily lives. We followed a regular math curriculum with the intent of not only learning a discipline, or coming away with a product, but we also agreed that HOW we gain this knowledge was very important.

Suddenly the students opened up creative math personalities that realized both the mechanics of a conventional traditional discipline while at the same time acknowledged how they gained the knowledge. During this 8 week practicum we began to develop both internal and external trust, even if we didn't know the right answer immediately. Or if we knew the right answer but didn't understand the mechanics of how we got it, we learned to trust the risky process of sharing our ideas. We came to feel that if we got beyond our own judgments or fear of the judgments of others, we would learn something, and usually much more than what was set out by the curriculum guide. We were developing internal authority in the subject area of mathematics.

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COMPARISON OF TEACHER AND PUPIL PERCEPTIONS OF THE LEARNING ENVIRONMENT IN MATHEMATICS CLASSES

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Comparisons were made between teacher and pupil perceptions of eight facets of the mathematics classroom environment difficulty, speed, inquiry, diversity, satisfaction, competitiveness, formality and goal-direction Questionnaires measuring classroom environment were administered in 60 junior high school mathematics classes to 1338 pupils and 32 teachers Teacher perceptions tended to be more positive than pupil perceptions statistically significant differences were found on competitiveness, formality, diversity, speed and difficulty Grade level, ability level, sex ratio and teacher gender had little effect on teacher-pupil differences

In most school systems around the world, mathematics instruction takes place in a group context Even if instruction tends to be more individual than collective, it does not occur in isolation but rather in interaction with a teacher and in the presence of other pupils who are engaged in similar learning activities Therefore, the classroom learning climate must be considered as an influential force when discussing both attitudes towards mathematics and mathematics achievement, two topics of considerable concern to educators and laymen alike

The learning environment, shaped by pupils, the teacher and the curricular materials, has the potential of influencing pupil attitudes, thereby affecting motivation to learn and subsequent learning Several studies have clearly pointed to a significant correlation between achievement in various school subjects and the learning environment (Anderson, 1973, Fraser, 1986, Hofstein & Ben-Zvi, 1980) In one study of mathematics classes, conducted by O'Reilly (1975), the learning environment was found to explain 67% of the variance in average class achievement scores

Taking into account these findings, it would seem that teachers would be more effective if they could accurately perceive not only individual pupil attitudes towards the study of mathematics but group opinions and the collective environment as well. The purpose of the present study is to determine to what extent mathematics teachers' perceptions of the learning environment coincide with those of their pupils.

Assuming that the classroom environment can be conceived as composed of various facets, then one must ask which facets do teachers tend to perceive correctly and which incorrectly. Studies on the determinants of the learning environment in different school subjects have shown grade level (Fresko & Ben-Chaim, 1986, Fresko, Carmeli & Ben-Chaim, 1988, Randhawa & Michayluk, 1975, Shaw & MacKinnon, 1973, Welch, 1979), sex ratio (Walberg & Ahlgren, 1970), and teacher gender (Lawrenz & Welch, 1983) to influence pupil perceptions. In the present study, an attempt will be made to establish whether these variables plus class ability level influence the degree of teacher accuracy in perceiving the classroom learning climate.

Several studies comparing teacher and pupil perceptions have been reported on by Fraser (1986). Findings from these studies indicate that teachers tend to have a more positive evaluation of the classroom climate than their pupils. Such results are consistent with research in other settings which indicate that individuals who have more authority and responsibility in a setting tend to hold more positive attitudes towards the setting (Moos, 1979). Accordingly, we hypothesize that mathematics teachers will also regard classroom climate in a more positive light than their pupils.

Methodology

Procedure

A learning environment questionnaire was administered to pupils and teachers in 60 junior high school mathematics classes (Grades 7 to 9) Administration took place approximately three months after the start of the school year in order to allow the classroom environment to crystallize In total, 1338 pupils and 32 mathematics teachers in 6 schools completed the questionnaire Five schools were situated in economically disadvantaged areas and one school, which was larger than the others, was located in a middle class neighborhood The teachers, who completed the questionnaire simultaneously with their pupils, were asked to respond to items as they thought their pupils would respond

The Questionnaire

A learning environment questionnaire for mathematics classes was employed in which eight properties (sub-scales) were examined difficulty, inquiry, satisfaction, speed, diversity, competitiveness, goal-direction and formality The development and one application of this questionnaire was described at PME-10 by Fresko and Ben-Chaim (1986)

Questionnaire items took the form of general statements referring to the entire class and a 4 point Likert-type response scale was provided from 1- "It never happens in my class" to 4- "It always happens in my class" Sub-scales were composed of 2 to 5 items Individual sub-scale scores were calculated by averaging responses on all relevant items Class averages were computed from the individual pupil mean scores

Results

Use of MANOVA indicated that pupil and teacher questionnaires were significantly different ($p < .0001$). The results of a series of t-tests for dependent samples in which teacher and pupil perceptions were compared by sub-scale are presented in Table 1. As can be seen, teachers differed significantly from their pupils on difficulty, speed, diversity, competitiveness and formality and concurred with their pupils on satisfaction, inquiry and goal-direction. In general, teachers rated the mathematics learning environment as more difficult, slower paced, more diverse, more competitive and less formal than their pupils.

Matched t-tests were performed by sub-scale after the sample was cross-sectioned separately by each of the following class traits: grade level (Grades 7, 8, 9), ability level (high and intermediate), sex ratio (1/3 or less boys, 1/3 to 2/3 boys, and 2/3 or more boys) and teacher gender. Results showed that while these variables tended to differentiate among different learning environments, for the most part they did not affect the accuracy of teacher perceptions. For example, it was found that pupils perceived the learning environment as less satisfying as grade level increased, and teacher perceptions concurred with this view.

A few exceptions did exist with respect to this general trend. For example, male teachers tended to err in fewer areas than female teachers, and teachers in classes having predominantly either boys or girls tended to err less than those in mixed classes.

In all comparisons, the two areas in which large and consistent differences existed between teacher and pupil perceptions were competitiveness and formality. Teachers always rated formality considerably lower and competitiveness considerably higher than their pupils. Moreover, the one area in which teacher perceptions always coincided with those of their pupils was that of satisfaction

Table 1

Teacher and class means, t-values and significance levels for dependent samples, by sub-scale.

Sub-scale	Teacher means	Class means	t	significance
Difficulty	2 31	2 21	2 23	015
Inquiry	2 98	2 92	76	224
Satisfaction	2 87	2 91	- 76	224
Speed	2 07	2 20	-2 53	007
Diversity	2 01	1 89	2 54	007
Competitiveness	2 93	2 38	5 90	000
Goal-direction	3 22	3 25	- 56	289
Formality	2 29	2 61	-4 42	000

Summary and Discussion

The mathematics teachers in this study were not always accurate in their assessments of pupil perceptions of the learning environment. Moreover, the class traits and teacher trait examined here had little consistent effect on their accuracy.

In accordance with prior research results, it was hypothesized that teachers in this study would hold a more positive view of the classroom environment. With respect to diversity, formality and speed, this contention was clearly supported. Teachers perceived the environment as more diverse, slower-paced, and less formal than their pupils. Teacher perceptions of the environment as more competitive may also be construed as a more positive evaluation. Given that five of the six schools have disadvantaged pupil populations whose mathematics achievements are low, it seems as if the teachers perceived a more highly-motivated environment than did their pupils.

Only with respect to difficulty can it be claimed that teacher perceptions differed from their pupils in a negative direction, i.e. they perceived the environment as more difficult. Perhaps taking into account the relatively low mathematics achievements of their pupils, they assumed pupils would perceive the mathematics learning environment as more difficult than they actually do.

Through the application of an instrument similar to the one employed in this study, mathematics teachers could become more aware of pupil perceptions of the learning environment in their classes. Strategies could then be employed to improve the environment and also increase the chances for effective learning to occur.

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NUMERACY WITHOUT SCHOOLING

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Developing new forms of representation is a major activity in mathematics. Perhaps for this reason, many educational practices are based on the assumption that learning new mathematical representation is the route to understanding concepts. This paper briefly considers a psychological theory coherent with this position and some evidence on the role of mathematical representation in cognition. This evidence indicates that numeration systems may be understood without writing but that learning arithmetic representations may result in the re-organization of arithmetic problem solving behavior. Caution is, however, called forth; experimental work is still needed for causal connections to be accepted.

Russian psychologists like Vygotsky and Luria have proposed that much of cognitive development can only be understood if we take into account the systems of representation (like language) which are socially transmitted and evolve from means of communication to means of organization of one's own behavior. In order to produce a picture of what cognition may look like without language, their investigations on the role of language in the organization of behavior included work with subjects whose linguistic processes are somehow limited. Luria and his co-workers studied both young children acquiring language and aphasic patients, investigating the impact of their limited linguistic abilities upon performance in cognitive tasks. Like language, mathematics provides representational tools which can be used both in interpersonal communication and in the organization of behavior. Luria (1969) attempted to use in the study of the role of mathematical representations the same approach he had profitably used with language--namely, the investigation of adults who had learned mathematics but later suffered some sort of brain injury. On the basis of this approach, he distinguished between the "*elementary functions of number*" and the "*semantic structure of number*". The *elementary functions of number*, according to Luria, include the abilities to count and designate a concrete quantity by a number, to perform computations (especially addition) with small (one-place) figures, and to carry out direct comparisons between quantities. The *semantic structure of number* includes composing two- and three-digit numbers, which are interpretable through socially conventioned units (see Gal'perin and Georgiev, 1969) such as "tens" and "hundreds" and the ability to compute with larger figures. According to Luria, the semantic structure of number appears to be mediated by the left hemisphere--i.e., mediated by symbolic processes since lesions to this hemisphere result in pathologies in this type of mathematical ability. In contrast, the elementary functions of number are retained in left-hemisphere damaged patients--an observation which attests to its independence from more complex symbolic processes. Luria's approach to the study of the role of symbols in the organization of mathematical behavior, though useful and instructive, is of limited applicability since few patients can be found for analysis. Moreover, results obtained with this method are sometimes difficult to interpret. If a particular ability remains intact despite lesions in the left hemisphere, it is possible to accept its independence from left-hemisphere functions. However, the observation of impaired functioning in patients with left-hemisphere damage sheds very little light upon the specific way in which mathematical representation affects the organization of mathematical cognition.

This paper attempts a different approach to the same question of how mathematical representations influence mathematical behavior. It reviews studies (some of which were not reported hitherto) on how pre-school children differ from school-age children and how uninstructed adults differ from schooled adults when solving mathematical problems. It starts from the consideration that some aspects of mathematics are met by people in everyday transactions and have become incorporated in natural languages--for example, comparisons of quantities (more, less, greater, smaller, higher, lower etc.), numeration systems (especially in their oral form), and measures of different types of quantities (including monetary systems and measures of length, weight, area, volume etc.). Other aspects are much less frequently encountered outside school--like mathematical sentences with symbols for operations (for example, 120×5), representations of unknowns (for example, $120 + y$) or representations of one variable as a function of another (for example, $y = 5x$). Despite the fact that many people must in their everyday lives deal with these concepts, everyday life does not present opportunities for people to learn the same type of representations which are used in school when they deal with these concepts. Furthermore, natural language has its own ways of classifying and representing

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mathematical situations which do not map onto the mathematical representation in simple ways. It is likely that some re-organization of behavior must occur so that situations well understood in everyday life can be represented in the conventional mathematical fashion. For this reason, a better understanding of how everyday representations in natural language are re-organized under the influence of formal representations learned in school seems very important for understanding how mathematical concepts develop.

If we want to reach even preliminary conclusions about the influence of learning mathematical representations upon mathematical cognition we must consider both the differences between pre-school and school-age children and the differences between unschooled and schooled adults. If one looks only at children, age and school-learning are confounded; differences between pre-school and school-age children could either result from specific learning experiences at school or from general cognitive development. If one looks only at adults with varying degrees of schooling, the role of mathematical representations in the organization of behavior is difficult to interpret because learning new forms of representation when conceptual understanding is well advanced may have different consequences from learning them at an earlier phase of development.

This review will concentrate on two aspects of mathematical knowledge and how they may relate to conventional representations transmitted in school: numeration systems and arithmetic operations.

1. ***Knowledge of numeration systems.*** Luria (1969) relates the understanding of the semantic structure of number to its writing within a place-value system. On the basis of his clinical studies and his review of the literature, he argued that designating a set by the number of elements after counting is preserved in left-hemisphere lesioned patients because all numbers are treated as single-digit numbers. However, his patients with left-hemisphere damage had great difficulty in reading or writing numbers with two or more digits, in reading arithmetic sentences, and in adding two- or more-digit numbers. Luria emphasizes how these difficulties in reading and writing are associated with the other difficulties in understanding number.

A different way of evaluating the relationship between mathematical cognition and knowledge of written numerals is to look at what pre-school children and illiterate adults understand or fail to understand about the socially conventioned units in the numeration systems they use. Saxe (1981), Saxe and Posner (1983), and Carraher (1985) have investigated knowledge of numeration systems among illiterate adults and also changes in this understanding as schooling takes place. In summary, Saxe and Posner found a re-organization of a non-base numeration system into a base-system among the Oksapmin both in children as schooling was introduced into the village and in illiterate adults as commerce became an integral part of their everyday activities. Thus, written representations of numeration systems are not necessary for the semantic understanding of number to be obtained. Carraher (1985) found further evidence favoring the independence between understanding properties of numeration systems and knowing how to write numbers. About one third of the Brazilian pre-school children and most of the illiterate adults she interviewed were able to understand basic properties of numeration systems despite the fact that they had not yet learned the written notation used in their culture. On the basis of these studies, it is possible to conclude that conventional written representations are not necessary for the understanding of the semantic structure of number; both the exposure to systematic teaching and everyday practices involving money may promote the development of these more complex numerical functions.

2. ***Knowledge of arithmetic operations.*** Several studies have been carried out on how children's problem solving behavior changes with age but no explicit consideration was given in these studies to the role of instruction and the introduction of signs to represent arithmetic operations. Recently, Carraher and Bryant (1987) and Carraher (1988) analyzed the difference between solving problems correctly with any representation the individual may choose and solving problems with the appropriate school representation. Pre-school children, who had not yet learned arithmetic representations, differed from first grade children, who had been taught these representations, both in their rate of

correct responses and in the strategies they chose for solving problems. Similarly, adults enroled in literacy programs, who had not yet learned arithmetic representations, and adults enroled in third and fifth grade classes differed in rates of correct responses and strategies in problem solving. Unschooled adults were better than school children at solving problems but they used in their solutions the arithmetic operations suggested by everyday interpretations of the situations rather than those operations which would be used according to the arithmetical sentences taught in school. For example, they chose addition to solve missing-addend problems (of the form $a + x = b$) and subtraction to solve missing-minuend problems (of the form $x - a = b$). These solutions indicate a representation of the problems in terms of their linguistic representations in everyday life rather than in terms of the arithmetic representations used in school (which would have been $b - a = x$ for the missing addend and $b + a = x$ for the missing minuend problems). In contrast, children and adults, after some schooling, tended to use solutions indicative of an arithmetic representation of the problem situations. This pattern of results suggests that the understanding of arithmetic operations is re-organized under the influence of schooling--possibly as a result of learning of new mathematical representations. However, in order to make assertions of a causal nature, experimental evidence is necessary.

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STRATEGIES AND ERROR PATTERNS IN SOLVING ROTATION TRANSFORMATION

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Based on a result of two years project, Taiwan pupil's correct strategies and error patterns were classified. However, features of the approaches used by those students in solving rotation items seem to be qualitatively different in nature. As a result of those students performance on the written tests and the interviews, this paper examines the reasons of students' errors and explores the ideas of each correct strategies. Moreover, the relative role of visual approach and analytical approach are discussed.

INTRODUCTION

Visual and analytical thinking have been indicated as important features during children's Mathematics learning career (Bruner, 1963; Krutetskii, 1976; Presmeg, 1985; Eisenberg and Dreyfus, 1986). Bishop (1988) noted that the notion of visualisation is an important focus in the education process. According to the relative role of verbal-logical and visual-pictorial components of a pupil's mental activity, Krutetskii (1976)determined the type of mathematical giftedness of an individual into analytic, geometric, and harmonic. Studying the responses on the written tasks which could be solved by both a visual and an analytical approach of some experts in Mathematics: research mathematicians, high-school mathematics teachers, and third year university students, Eisenberg and Dreyfus (1986) classified them as being analytical, visual or mixed.

In the CSMS works (Hart, 1981), Küchemann (1981) described levels of understanding of rotation in terms of pupil's performance on the rotation tests. Their facilities varied considerably, depending on the position of the centre of rotation and the slope of the object, but some lesser influences depending on the object's complexity, and the presence or absence of a grid. For example, children at level 1 could sketch the image of flag which was vertical or horizontal, and given that the centre was on the object. Children at level 2 could sketch the image of triangle given that the centre was on the object and recognize that the centre of rotation had to be equidistant from a pair of corresponding points on the object and its image. For level 3 children, they could sketch the image of flag which the centre was directly underneath the end-point, and tried to construct mentally a line from the centre to the object. Children at level 4 could sketch the image of flag which was vertical and the line from the centre to the base-point of the flag is horizontal but with a grid, and were beginning to recognize that the angle between the lines from the centre to corresponding points on the object and its image is equal to the angle of rotation. Then, children at level 5 could rotate any objects through a quarter turn regardless of its slope or its position relative to the centre. Küchemann (1981) also described some approaches which seemed to be used by English

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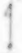
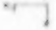
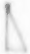
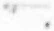
children. To rotate an object, a child may build up the image by locating all (resp. some) crucial points of the object first, and then adding the rest portions. The child is using the analytic (resp. step-by-step) strategy.

Based on the CSMS works (Hart,1981), an adaptation of the CSMS methodology and rotation test (Küchemann,1981) were used in Taiwan. In order to make sure that students could recognize some ideas of rotation transformation, a practical manipulation of rotation transformation were presented by researchers before the testing. The written test (Lin and Chin, 1987) in this study were given to 2811 students aged 13 to 15 years, those samples will be expected to be representative of the Taiwan population of that age.

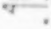
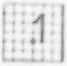
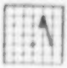
Taiwan data showed that the key features which would affect children's performance were the position of the centre of rotation (item 1 vs. item 4) and the complexity of the structure of object (item 1 vs. item 2.3). To English students the key features were the position of the centre of rotation and the slope of the object. As appears in Table 1.

Table 1

Item		Facility N=2811	
Sketch the image of object		Taiwan(%)	CSMS(%)
1		69	85
2		38	74
3		40	67
4		31	44

To a lesser extent, the features for Taiwan students were the slope of the object (item 6 vs. item 7) and the presence or absence of a grid (item 5 vs. item 6).but for English students were the complexity of the structure of object and the presence or absence of a grid. As shows in Table 2.

Table 2

Item		Facility N=2811	
Sketch the image of object		Taiwan(%)	CSMS(%)
5		31	44
6		29	27
7		20	18

Analyzing the responses of Taiwan students in the written tests, their correct strategies and error patterns which could be identified were classified. Some examples appear in Table 3 and Table 4.

Table 3
Correct strategies N=2811

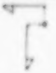
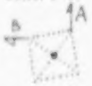
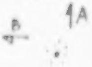
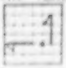
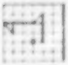
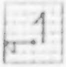
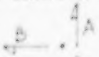
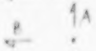
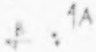
Strategy	Response	Incidence(%)
	Sketch the image of object	
'Connecting'		24
	Draw the centre of rotation	
'Geometrical construction'		0.8
'Trial and error'		4

Table 4
Error patterns N=2811

Response	Incidence(%)
Sketch the image of object	
	11
	19
	10
Draw the centre of rotation	
	[illegible]
	[illegible]
	22

This work aims:
(1) to investigate the reasons for each kind of errors.
(2) to explore the underlying ideas for each correct strategies.

Methodology

1. Sample

To each correct strategy and error pattern we selected some students whose performance were identifiable for the interview. A total sample of 70 students would be interviewed.

2. Interview

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Two kind of items: the sketching image and the drawing centre of rotation will be concerned in the interviews. The procedure was carried out by following steps.

A. Developing interview items

The rotations under consideration are fixed to be a quarter turn anticlockwise. It seems that a successful performance on the rotation consists of a global view i.e. recognizing the location of image globally and some specific features i.e.recognizing:

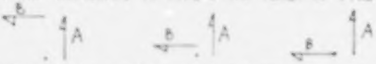
(1) the angle (90°) between the lines from centre to one pair of corresponding points on the object and its image.

(2) the equidistance of the lines from the centre to one pair of corresponding points.

(3) the mutually perpendicular between the object and its image.

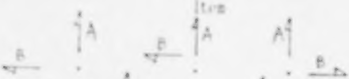
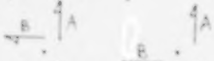
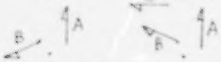
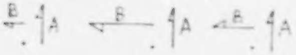
(4) the congruence of the object and its image.

Regarding each features, a set of items will be developed to examine children's recognition about it. For instance, is flag B the image of flag A ? why ?



These three items would be used to examine students recognition about 'the equidistance'. The items developed to examine other features are outlined in Table 5.

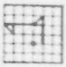
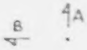
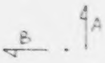
Table 5

Feature	Item
'the global feature'	
'the angle'	
'the mutually perpendicular'	
'the congruence'	

B. Error analysis based on written test

Supposed reasons for each error patterns were organized before the interviewing. Some instances appear in Table 6.

Table 6

Error pattern	Supposed reason
	.recognize 'the global feature', 'the congruence','the mutually perpendicular' and 'the angle'. .fail to recognize 'the equidistance'.
	.recognize 'the global feature','the congruence', and 'the mutually perpendicular'. .fail to recognize 'the equidistance' and 'the angle'.
	.recognize 'the global feature', 'the congrunence', 'the mutually perpendicular',and 'equidistance'. .fail to recognize ' the angle'.

C. Interviewing

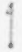
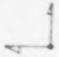
On the interview, each student was shown the test paper completed earlier, and asked to explain the answers of some items they had completed. In addition, according to student's error patterns and correct strategies, they were asked to do some items which were developed for the interviews. The purposes of the interviews were to examine or revise the supposed reasons of each errors patterns as well as to explore the underlying ideas of each correct strategies.

RESULTS

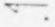
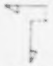
For the convenience of analysis, we use triplet (, ,) to represent test items, childeren's responses, and facility/incidence respectively. Some students' explanations of both correct strategies and error patterns appear as follow.

1. Children's ideas of correct strategies

The fundamental reasons of students' correct strategies are founded on their recognition of 'the global feature' and 'the congruence'. The maturity of these reasons might be qualitatively building up by recognizing other specific features and depending on the facility of items. This might be explained by the following interview data.

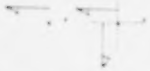
A. ( ,  , 69)

Students explanation	Analysis
. Just like drawing a quarter of circle, a quarter turn means 90° or right angle.	. recognize 'the angle' only by drawing a circle.
. anticlockwise, so the image of flag must lay down on the left hand side of the object.	. recognize 'the global feature'.
. a flag is a flag, it is unchange in size and shape after rotation.	. recognize 'the congruence'.

B. ( ,  , 24)

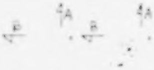
In the interviews, the students who used this strategy recongized 'the global feature' and 'the congruence' as showed in previous case. In addition, they also recognized 'the angle' and 'the equidistance'.

Students explanation	Analysis
. since the centre is not on the flag, so I need a pivot to rotate it, hence, I construct a dotted line which connect between the base-point of object and the centre, then I can rotate it easily.	.stabilization.
. a quarter turn means 90° ,so the angle between the dotted lines are at right angle.	.recognize 'the angle'.
. like drawing a circle, all radia of circle must be of equal length	.recognize 'the equidistance'.

C. ( 4)

The students who used this strategy recognized all features of rotation. However, their ideas got directly from their geometrical knowledge. In the interviews, they explained :

"..... I look the centre '.' as an origin in the plane, since a quarter turn means 90° or orthogonal, so I draw two lines x-axis and y-axis through the origin at right angle...."

D. ( 4)

The students who used this strategy reflected a 'trial and error' strategy. They tried to coordinate 'the angle' and 'the equidistance'. For instance, they said:

"I try many times to find the centre of rotation in my mind, ..., and since the angle between the lines from the centre to corresponding base-points on both flags is equal to 90° as well as the lines being of equal length."

E. (0.8)

The students who used this strategy not only recognized all features of rotation but also reflected their rich geometrical knowledges. In the interview, one of them explained:

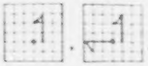
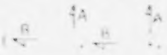
"because the angle between the lines from the centre to corresponding base-points on both flags is equal to 90°, so I consider it's maybe a right triangle. ...., since the lines being of equal length, hence I relate it to the isosceles triangle, and then suddenly I figure out that a regular square can be constructed by four of these isosceles triangles, therefore, I suppose that the common vertex of these triangles might be the centre."

Subsequently, I asked him to confirm his conjecture. He said:

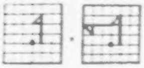
"because it's a isosceles triangle so both sides [the lines from the centre to corresponding base-points on both flags] are equal, and they [isosceles triangles] for a regular square, so the angle [angle between both sides as he noted before] must be equal to 90°."

2. Reasons of error patterns

Almost all students in the interviews recognized 'the global feature', 'the congruence' and 'the mutually perpendicular'. However, they failed to recognize 'the angle' or 'the equidistance'. This might be explained by the following interview data.

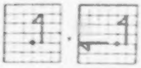
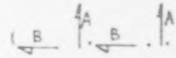
F. ( 10), ( 25)

Students explanation	Analysis
. anticlockwise, so the image must lay down on the left hand side of the object.	. recognize 'the global feature'.
. a quarter turn means 90° or right angle, then the image and object must be mutually perpencicular.	. fail to recognize 'the angle'.
. since object is vertical, so the image must lay flatly.	. recognize 'the mutually perpendicular.
. the centre give me a hint, so I sketch the image flatly base on that point.	. fail to recognize 'the equidistance'.
. no matter how you rotate a flag it is still a flag.	. recognize 'the congurence'.

G. ( , 19)

Students who made this kind of error could recognize 'the global feature' , 'the congruence', and 'the mutually perpendicular', but failed to recognize 'the equidistance'. However, they did recognize 'the angle'. They said:

> "... the angle between lines from the centre to corresponding base-points on the object and its image is at right angle...., and the base-point of the image can lay on any position of the line from the centre to the base-point of image. "

H. ( , 11) , (

. 35)

Students who made these kinds of error recognized 'the global feature', 'the mutually perpendicular', and 'the congruence', but failed to recognize 'the angle'. However, they did recognise ' the equidistance'. In the interviews, they said:

> "to rotate a flag in a quarter turn anticlockwise, just like drawing a circle, ". " means centre of the circle and then all its radia must be of equal length..."

Therefore, it is evident that the reasons of each error patterns are confirmed to be the same as those supposed reasons.

DISCUSSION

Considering the relative role of visual-pictorial and verbal-logical components of a pupil's mental activity, Taiwan students' approaches seem to be different in nature. Because 'the global feature', 'the mutually perpendicular' and 'the congruence' are visual in nature, students could recognize them by looking at the practical manipulation or doing pratice question. Hence, if a student recognized these features and relies much on them to solve rotation items, then , we say that he uses visual approach on rotation tasks. In other words, this approach is characterized by the predominance of a strong visual-pictorial over a weak verbal-logical one. Therefore, the students who used this approach could only rotate the objects by their limited visual experiences.

However, in solving a rotation item that the position is not on the object or the structure of the object is complex, students have to recognize some specific features of rotation which are analytical in nature e.g. 'the angle'and 'the equidistance' through analysing the features of rotation transformation. If a student recognizes these features and relies much on them to solve rotation items, then, we say that he uses analytical approach on rotation tasks. In other words, this approach is characterized by the tendency of a strong analytical idea which is based on visual experiences. Taiwan data showed that many students reflected their analytical approach on solving rotation tasks either explicitly or mentally. In particular, a small sample of students could apply their geometrical knowledge to amplify their ability in solving rotation transformation, but there were no evidences for English students.

Additionally, if a student recognizes the features of visual approach but fails to recognize either 'the angle' or 'the equidistance' features of analytical approach, then we say that he uses semi-analytical approach on rotation tasks.

Over all, Taiwan students did exist these three approaches which were strictly different in nature. Their spontaneous responses about rotation transformation were in a visual situation. Therefore, the students who used visual approach could only solve easier items in rotations. They must coordinate both visual approach and analytical approach for solving more difficult items. In order to grasp all specific features it is necessary to develop their analytical problem solving ability.

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A SOLO MAPPING PROCEDURE

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Summary

The work described in this report puts forward a technique for analysing responses in mathematics. After setting the technique in a theoretical framework based on the SOLO Taxonomy (Biggs & Collis, 1982), the report demonstrates how it can be used to distinguish among various levels of students' responses to mathematics items.

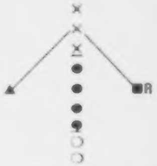
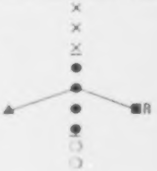
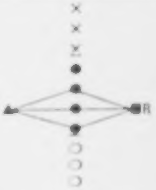
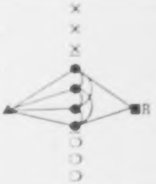
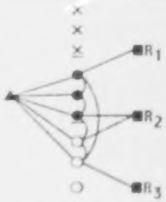
Theoretical Orientation

During the current decade a number of similar models have been put forward to explain the increasing complexity of cognitive functioning observed as children mature and become adults (Biggs & Collis, 1982; Case, 1985; Fischer, 1980; Halford, 1980). The model devised by Biggs and Collis (SOLO Taxonomy) has been used quite extensively for evaluation purposes in school content areas (e.g., Collis & Davey, 1986; Collis, Romberg & Jurdak, 1986) and it is this model which provides the infrastructure for the mapping procedure described here.

The SOLO model enables an answer to a particular question to be classified by the way in which it is structured. According to the model there are five basic categories within the concrete symbolic mode of functioning (i.e., approximately 7+ years to 15+ years). Figure 1 (adapted from Biggs & Collis, 1982, pp. 24-25) summarises the key elements of the five base categories. Column (i) gives the name of the category; column (ii) refers to the demands made on the individual's working memory; column (iii) outlines the level of logical operations shown by a response in a category; column (iv) summarises the consistency and psychological closure characteristics of each category and, finally, column (v) sets out a metaphor for the model.

Figure 1

SOLO MODEL AND RESPONSE STRUCTURE

(i) SOLO description	(ii) Working memory capacity required	(iii) Logical operations involved	(iv) Consistency and closure	(v) Response Structure Cue Response
Prestructural	*Minimal:* cue and response confused	Denial, tautology, transduction. Bound to specifics	No felt need for consistency. Closes without even seeing the problem	
Unistructural	*Low:* cue + one relevant datum	Can "generalize" only in terms of one aspect	No felt need for consistency, thus closes too quickly: jumps to conclusions on one aspect, and so can be very inconsistent	
Multi-structural	*Medium:* cue + isolated relevant data	Can "generalize" only in terms of a few limited and independent aspects	Although has a feeling for consistency can be inconsistent because closes too soon on basis of isolated fixations on data, and so can come to different conclusions with same data	
Relational	*High:* cue + relevant data + inter-relations	Induction. Can generalize within given or experienced context using related aspects	No inconsistencies within the given system, but since closure is unique so inconsistencies may occur when going outside the system	
Extended Abstract	*Maximal:* cue + relevant data + inter-relations + hypotheses	Deduction and induction. Can generalize to situations not experienced	Inconsistencies resolved. No felt need to give closed decisions --conclusions held open or qualified to allow logically possible alternatives (R_1, R_2, or R_3)	

Legend: × = irrelevant or inappropriate; ● = relevant; ○ = relevant, abstract and hypothetical (Adapted from Biggs and Collis, 1982, pp. 24-25)

Although all facets of Figure 1 are important in the analysis of learning, it is column (v) which is the focus of our interest in developing a mapping procedure for the analysis of responses to mathematics items. Column (v) has been expanded at each of the five levels to show the data processing which is presumed to take place prior to the response being given. The following symbols are used in subsequent diagrams:

✖ - inappropriate or incorrect data, concepts, processes or strategies,

▲ - data given with the potential to cue a response,

● - concepts, processes and/or strategies expected as part of the 'Universe of Discourse' and thus of the understanding of the question,

○ - abstract concepts, processes and/or strategies within the 'Universe of Discourse' but additional to those expected as part of the understanding of the question,

■ - responses.

These symbols are used to expand the metaphor in Figure 1 by separating the Cues into the Question which is asked and the Data which are given in the problem. The central part of the map contains the Concepts and/or Processes which are employed in obtaining a solution. Finally the Response component of the map is split into Intermediate and Final parts.

Mapping a Problem

The key elements of the mapping procedure will be illustrated by solving a typical mathematics problem set in the superitem format described by Collis, Romberg & Jurdak (1986). This format requires that the item has a stem which provides basic information and four questions which are designed with the criteria for the four higher levels of SOLO responses in mind. The first question requires only a unistructural level of response; the second, multistructural; the third, relational and the fourth involves demonstrating the ability to make an extended abstract response.

Stem: $2^3 = 2\times2\times2 = 8$

$3^2 = 3\times3 = 9$

$\Delta^2 = \Delta\times\Delta$

and $\Delta^3 = \Delta\times\Delta\times\Delta$

Question 1: Find the value of 4^2.

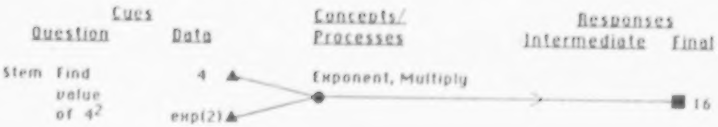
Question 2: Find the value of 5^4.

Question 3: What is the value of 'Δ' if $(\Delta+1)^3 = 64$?

Question 4: If $(c+a+1)^3 = 512$, what pairs of whole number values can 'c' and 'a' take between 0 and 7?

An item must have an orienting stimulus which alerts the individual to the 'Universe of Discourse' in which the problem is set. In the case of the item under discussion this is achieved by use of a separate stem but in many cases the nature of the symbolism in the question is sufficient. Let us take the questions one by one and map a path which an individual might follow to obtain a correct solution.

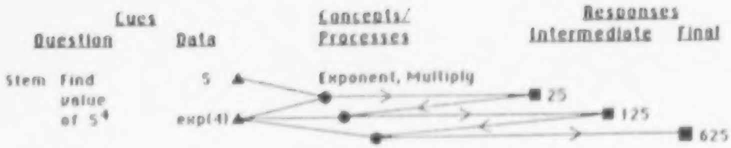
Question 1. Find the value of 4^2.



Having been alerted to the universe of discourse by the stem and the question, the individual is cued to begin by the question, 'Find the value of 4^2'. Next one finds the required information, 4 and exp(2), uses his/her concept of the meaning of 4^2, i.e., 4x4, and processes the information by multiplying to obtain the response.

Given familiarity with numerals and the four operations of arithmetic, the working memory space required is minimal and, as only one proposition (implying one operation) is involved, immediate closure is possible.

Question 2. Find the value of 5^4.


Cues
Question
Data
Concepts/
Processes
Responses
Intermediate
Final
Stem
Find value of 5^4
5
exp(4)
Exponent, Multiply
25
125
625

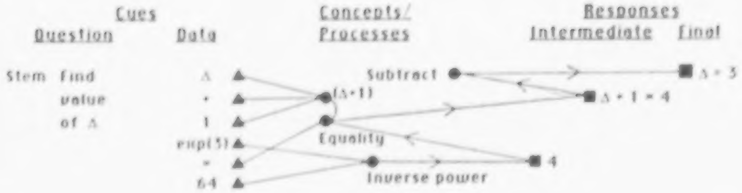
The path up to the concepts/processes section is similar to that in Question 1. However, at this point the student has to recall that the index shows the number of factors and when multiplying in sequence, it is necessary to monitor the number of factors used as one goes along. Working memory has to be used to monitor the multiplying process as well as to do the multiplication, thus more than a minimal amount is required and closure can only come after the sequence of multiplying is completed. In practice this means: selecting/forming the first proposition (5×5) and closing, 25; forming the next proposition (25×5) and closing, 125; forming the next proposition (125×5) and closing, 625; realising that the last closure is the end of the sequence required. The solution requires a series of disjoint operations while holding in mind how many need to be performed before stopping.

Question 3. What is the value of 'Δ' if $(\Delta+1)^3 = 64$?

A typical solution to this question would be the following:

"Take the cube root of both sides: $\Delta + 1 = 4$;

then subtract: $\Delta = 4 - 1 = 3$."

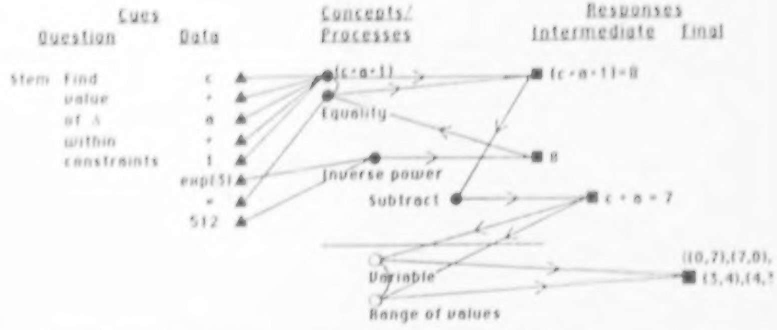

Cues
Question
Data
Concepts/
Processes
Responses
Intermediate
Final
Stem
Find value of Δ
Δ
+
1
exp(3)
=
64
(Δ+1)
Equality
Subtract
Inverse power
Δ = 3
Δ + 1 = 4
4

Again it is the concepts/processes section which distinguishes this level from the lower levels. The notion of treating 'Δ+1' as a unit and both the concept and the processing involved in the inverse power idea are quite sophisticated, as is the linking of these results together by means o equality and the subsequent solving of the equation generated. It can be seen that a significant working memory load is generated not only by the necessary calling-up of complex concepts but also by the processing of the mathematical operations involved. Monitoring these processes so that one does not get lost along the way also adds significantly to this load. Closure must be held off for a considerable period to allow all the necessary processing to be completed.

Question 4. If $(c+a+1)^3 = 512$, what pairs of whole number values can 'c' and 'a' take between 0 and 7?

A typical correct solution of this question would be the following:

"Take the cube root of both sides: $c + a + 1 = 8$; then subtract: $c + a = 7$; finally, consider all pairs of whole numbers which sum to 7: 0 and 7, 7 and 0, 1 and 6, 6 and 1, 5 and 2, 2 and 5, 3 and 4, and 4 and 3."


Cues
Question
Data
Concepts/
Processes
Responses
Intermediate
Final
Stem Find
value
of Δ
within
constraints
c
+
a
+
1
exp(3)
=
512
(c+a+1)
Equality
Inverse power
Subtract
(c+a+1)=8
8
c + a = 7
Variable
Range of values
(0,7),(7,0),
(3,4),(4,3

The extended abstract level is distinguished from the relational level by the introduction of relevant abstract concepts not included in the original data. In the present case this becomes necessary in the last stag of the solution process. The working memory load is much higher than at the previous level mainly because of the introduction of the abstract concepts from outside the data; the selection of the appropriate concepts and then the monitoring of the processes generated by them adds significantly to the demands. It is clear also that closure must be withheld while considerably more processing is carried out once the necessity for the introduction of 'outside' abstract concepts is accepted.

This item was designed with the four SOLO levels in mind. Some considerable trouble was taken to ensure that the questions would elicit a response at the required level. The procedure has also been used successfully with open-ended questions and in the analysis of errors which students make in responding to problems (e.g., Chick, 1988; Chick, Watson & Collis, 1988; Watson, Chick & Collis, 1988). The latter application has implications for remedial instruction. There is also an obvious application of the technique to the task analysis of problems before they are set for students.

Discussion

The SOLO model is a general model of cognitive functioning. Thus it would be expected that any technique which was a legitimate extension of this base would be able to distinguish clearly between the structural levels of responses for various mathematics problems. In addition, to be a useful extension in the present context it must enable a clear picture to be obtained of the different parts of the cognitive processing involved in going from question to answer, and allow the path followed by the reasoner to be drawn. These three considerations we argue are encapsulated in the technique described in this paper.

In conclusion the research potential of the technique should be mentioned. This potential rests on the basic underpinning of the SOLO Taxonomy itself in that the mapping procedure follows the response path actually traversed by the problem-solver; it does not rely on fitting the responses to some preconceived category. Two possibilities stand out.

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First, maps can be drawn for the responses of groups of students in different years of school for mathematical problems typical of those years with a view to observing the characteristic patterns of achievement and error. Second, for the researcher interested in the mechanism of problem-solving itself, the map of an individual's response enables a view of the process as it existed on the occasion of the testing to be used as starting point for both further analysis and empirical manipulation.

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LE DISCOURS JUSTIFICATIF EN MATHEMATIQUE: L'IMPLICATION DU LOCUTEUR SELON LA REPRESENTATION DU REFERENT

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Abstract

Children (9-11 years of age) give written argumentation to explain what must be done to win a mathematical game. All things being equal, the discourse varies according to the institutional situation of production: neutral uninvolved formal discourse in academic situations (FD) / natural discourse with the locutor's involvement and marks of certitude in game situations. The same opposition FD/ND is found in adults who have to solve a probability problem according as they have a "scientific" or "spontaneous" representation of the problem. Are there any consequences to be drawn as to the teaching of proof formulations in mathematics?

Pour rédiger une justification mathematique, il est d'usage, dans certaines conditions, d'utiliser un discours formel (DF), c'est à dire un discours neutre, impersonnel, décontextualise. Le locuteur fait appel à des enonces (théoremes, propositions) établis, reconnus par l'institution scientifique, il n'a pas besoin de s'impliquer dans son discours, ni de le prendre en charge; il surprendrait d'ailleurs en le faisant. L'eleve qui redige une demonstration pour l'institution scolaire se trouve dans un systeme de contraintes qui le conduisent à reproduire ce modele de Discours Formel.

Nous montrons dans ce papier, que la variation de certains paramétres de la situation entraine une modification notable du type de discours produit : celui-ci peut se rapprocher d'un discours argumentatif "plus naturel", méme lorsque l'objet de l'argumentation reste en apparence le méme: dans nos experiences, c'est un referent qui appelle un raisonnement scientifique.

Nous nous demandons dans ces conditions si les contraintes imposees à l'eleve ne constituent pas un risque de blocage lorsqu'il doit formuler une preuve en mathématique.

CADRE DES RECHERCHES

Pour nous, le discours argumentatif est le produit d'une conduite langagiere (C.L.) argumentative. Les C.L. repondent à des exigences fonctionnelles qui varient selon les situations : une C.L. est liee à une finalite qui fait partie de la representation elaboree par le locuteur : celui-ci traite l'ensemble des caracteristiques de la situation de production pour se construire une representation du discours qu'il doit produire à ce moment-là, à cet endroit-là, pour cet interlocuteur-là...

1) Bronckart (1985) décrit les situations à partir de configurations de parametres extra-langagiers qu'il classe selon trois "espaces":

- L'espace référentiel : il concerne les notions, relations, schématisations; il correspond aux représentations a-langagières élaborées ou mobilisées par le sujet.

- L'espace de l'acte de production : il est défini par les caracteristiques materielles de l'activite verbale : ecrit/oral; monologue/dialogue....

- L'espace de l'interaction sociale : il est caracterise par le lieu social (institution scolaire), le statut social de l'enonciateur et du destinataire (maitre-eleve), le but vise par la conduite langagiere...

Il est clair que pour nous ces trois espaces ne definissent pas des dimensions toujours totalement independantes (un elève explique rarement la solution d'un exercice à un camarade par ecrit); ils constituent neanmoins une base utile pour l'analyse des situations de production.

Dans les recherches présentees plus loin, nous nous appuyons sur des variations de la représentation de l'espace référentiel: a) soit provoquees par l'intermédiaire de variations du lieu social; b) soit invoquees en analysant les représentations des sujets.

2) Nous analysons dans nos recherches des discours produits en situation de raisonnement. Nous nous sommes pour cela appuyés sur les travaux de Grize qui distingue raisonnement naturel et raisonnement formel: ces deux types de raisonnement ayant leur construction propre et ne pouvant se deduire l'un de l'autre. Un certain nombre de critères

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permettent de distinguer les discours produits en situation naturelle (DN) des discours produits en situation formelle (DF): la nature des objets, le mode et la structure d'etayage, le degre d'implication discursive. C'est ce dernier critere repris par Mieville (1984-85) puis par Esperet, Coirier, Coquin, Passerault (1987) que nous avons retenu.

L'implication du locuteur dans son discours sera marquee par exemple par: la prise en charge du discours par le locuteur ("*je pense* que...."); la presence du locuteur dans son discours ("*je* choisis..."); la presence d'un interlocuteur personnalisé ("*Martin*, lui aussi....").

Alors que le caractere desimplique du discours sera repere par les formes impersonnelles (il semble, il y a...).

3) En faisant varier directement l'espace referentiel, Esperet et al (1987) ont fait apparaitre dans des productions ecrites d'enfants (7-14 ans) deux types de discours: l'un fortement desimplique, lorsque la question relevait d'un cadre scientifique et qu'il existait pour le locuteur une reponse correcte unique (ex: conservation du volume); l'autre, fortement implique, lorsque la question relevait d'un debat d'opinion et que plusieurs réponses étaient recevables (ex: autorisation de fumer à partir de 15 ans seulement).

Dans les deux recherches presentees ci-dessous, nous reprenons cette *hypothèse générale*: le niveau d'implication du locuteur dans son discours sera plus élevé lorsque celui-ci se fait une représentation "naturelle" du raisonnement à produire que lorsqu'il s'en fait une representation "scientifique formelle".

EXPERIENCE 1

(Cf Coquin, Patej, 1989)

La tâche: expliquer par ecrit ce qu'il faut faire pour gagner la "course à 20".

La "course à 20" (Brousseau, 1973) est un jeu qui se joue a 2 partenaires: chacun dit un nombre en rajoutant "1" ou "2" au nombre dit par l'autre. On commence à zéro. Le gagnant est celui qui reussit à dire "20" le premier.

Les variables

1ère variable: Le lieu social où se deroule l'experience:

- à l'ecole à l'heure du cours de mathematique; l'instituteur est present et participe à la presentation du jeu;

- dans un centre de loisirs, pendant les vacances d'ete à des horaires et des lieux habituels de jeu.

2ème variable: L'âge des enfants: 8-9 ans (CE2) et 10-11 ans (CM2). Nous avons au total 66 enfants repartis en 4 groupes.

Procédure expérimentale

- Presentation du jeu et exécution de 4 parties par binôme.

- Debat oral à propos du jeu et en particulier des strategies gagnantes.

- Reponse individuelle par ecrit à la question: "Que doit-on faire pour gagner la course a 20".

Traitement des données

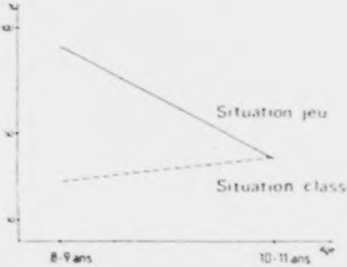
Sur chaque protocole nous avons releve comme marques d'implications:

- Les Prise en charge de l'enonce par le locuteur "*je pense* que ce jeu etait bien".

- Les Presences du locuteur: le locuteur est present dans son discours en tant qu'acteur: "*j*'ai mis le chiffre 17...".

- Les Presences d'un interlocuteur unique personnalise: "*Martine*, elle aussi, a gagne deux fois".

Resultats



Pourcentage d'enfants qui presentent au moins une marque d'implication.

L'hypothese principale est clairement verifiee: en situation jeu, beaucoup plus d'enfants s'impliquent dans leur discours qu'en situation classe. Deux interpretations sont à mon avis possibles: (1) L'enfant se construit une representation du *discours* qu'il juge comme recevable dans la situation ou il est. Il va donc depersonnaliser et desimpliquer son discours en classe, ou il pense qu'on attend un discours "formel", alors qu'il s'autorise differents types de marques d'implication en situation jeu. (2) L'enfant se construit une représentation du *referent* en fonction de la situation dans laquelle il se trouve: le jeu proposé lui apparaitra plus facilement comme un exercice mathematique s'il est en classe et comme un jeu reel, s'il est en vacances dans un centre de loisirs. Il choisira en conséquence le langage adapté à sa représentation du referent et non adapté directement à la situation.

L'allure de l'interaction "âge x situation" nous incite à pencher en faveur de la deuxieme interpretation: en effet, si la situation agissait directement sur la représentation du type de discours à produire, l'ecart entre DF et DN devrait s'accentuer avec l'âge puisque l'eleve maitriserait de mieux en mieux le type de discours qu'il doit produire en classe. Or, c'est l'inverse que l'on observe ici: malgre la difference de situation, les enfant de 10-11 ans reperent bien l'aspect mathematique de la tâche proposée et c'est cette représentation scientifique qui guide leur argumentation dans les deux cas; on n'observe aucune différence entre les discours. En revanche, les plus jeunes se laissent prendre au piege de la situation qui cree chez eux des representations differentes de la tâche demandee.

EXPERIENCE 2

(Cf Fayoux-Mesmin, 1988)

La tâche: résolution de trois problemes probabilistes simples (Maury, 1986 et Baillé et Maury,1988). Dans ces problèmes, on demande aux sujets de choisir entre deux possibilites de tirage, (par exemple 2 sacs de boules ou 2 roulettes), pour avoir le plus de chances de realiser un evenement donné. Dans tous les problemes, le choix est indiferent: l'évenement a la même probabilité d'être realisé dans les deux cas.

Les sujets: 88 étudiants de Deug II de Psychologie repartis en deux groupes. Ces sujets ont tous reçu un enseignement de probabilite au lycée, mais n'ont pas eu l'occasion de pratiquer depuis 20 mois, ce qui laisse le temps aux représentations scolaires de s'atténuer.

Les variables

1) **Le type d'exercice** (E; variable intrasujet à 3 modalités). Les exercices sont de difficulté croissante; les deux premiers sont relatifs à la quantification des probabilités, le troisième à la notion d'espérance mathématique.

2) **La formulation de la question** (F; variable intersujet à 2 modalités). Formulation Personnalisée (FP): "Quelle roulette *choisiriez-vous* pour avoir le plus de chances que..."/ Formulation Impersonnelle (FI): "Quelle roulette *faut-il* choisir pour avoir le plus de chance que..." Sinon, la formulation des exercices reste identique en tous points.

3) Ces deux variables manipulées expérimentalement, ne sont là que pour créer des variations de la représentation des problèmes posés aux sujets: **cette représentation étant la variable fondamentale** de l'expérience.

Procédure expérimentale

Les textes des trois exercices d'une même formulation sont présentés simultanément, toujours dans le même ordre aux sujets. Chaque sujet donne sa réponse et la justifie par écrit sur une feuille séparée.

Traitement des données.

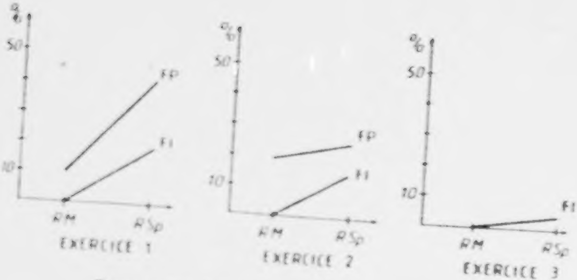
1 - Pour chaque sujet et chaque exercice, on détermine en fonction d'indices sémantiques et de l'argumentation utilisée, le type de représentation du problème: Représentation Mathématique (RM) lorsque le choix est fondé sur des arguments probabilistes pertinents (rapports, proportions) / Représentation Spontanée (RSp) lorsque d'autres arguments interviennent (nombre absolu de cartes, répartition des secteurs sur la roulette...).

2 - Pour chaque exercice on relève les marques d'implications (Cf expérience 1)et les formes impersonnelles.

Résultats

1) Les représentation spontanées sont plus nombreuses en FP qu'en FI et elles augmentent en fonction de la difficulté de l'expérience.

2)



Pourcentage de sujets qui presentent au moins une marque d'implication.

Dans tous les cas, l'implication du locuteur est plus forte en FP qu'en FI; ceci n'est qu'un effet d'amorçage de la question. Dans tous les cas egalement, l'implication du locuteur est plus forte lorsque la representation est spontanee que lorsqu'elle est mathématique, ce qui correspond à notre hypothese générale. En outre, pour les representations spontanées, l'implication decroit lorsque la difficulté de l'exercice augmente.

CONCLUSION

Il semble que les sujets laissent libre cours à leur implication lorsque la representation de la situation est moins scolaire et / ou lorsque la representation du referent est moins mathématique. En d'autres termes, la non-implication resulterait de contraintes dues au modèle scolaire à propos d'un probleme mathematique. Ayant vérifié des effets similaires pour d'autres marques (connecteurs, marques formelles par exemple), nous nous demandons si ces contraintes ne constituent pas un frein à l'apprentissage de la formulation en mathematique et si un assouplissement du modèle ne faciliterait pas cet apprentissage.

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QUALITATIVE AND QUANTITATIVE TIME REASONING IN CHILDREN

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ABSTRACT

Our proposal concerns the development of temporal inferences when the information is given in the form of a hypothetical situation. As far as the qualitative and quantitative aspects of reasoning are concerned, our hypothesis is that the presence of numerical givens make the task resolution easier because this verbal information (quantitative) lends to the construction of algebraic integration rules. Seven problems were presented to fifty seven children of 3 age groups: 6 years, 9 years and 12 years old. The seven problems were of two types, qualitative and quantitative, thus the children were presented fourteen experimental situations. The results show a facilitative effect of the quantitative wordings for every age and every type of problem (equality, inequality). Although, 6 year old find the qualitative form for the t2= type problems most difficult.

INTRODUCTION

Temporal reasoning, although very complex, does not seem to be analyzed sufficiently at the present time, especially concerning its double aspect of having qualitative and quantitative features.

The traditional experimental paradigm, concerned with the nonkinematic frame (duration, temporal orders relations), (Levin, 1977, 1982; Levin, Goldstein & Zelniker, 1984; Levin, Israeli & Darom; 1978 ; Levin, Wilkening & Dembo, 1984; Montangero, 1977, 1984 ; Richie & Bickhard,1987), consists of presenting a child information concerning the temporal order of two displayed lamps that are each switched on and off. This paradigm has been based on a set of manipulated objects (e.g. the subject is asked to judge the relative duration of two lamps are switched on and off).

These studies have pointed out two difficulties for the young child. One is that the most difficult comparisons to make concern duration and the other is that reconstituting duration from the initial succession order (when the final succession order is simultaneous) is of a superior difficulty than the inferred relative duration from the final succession orders (when the initial succession order is simultaneous). Despite the fact that the young child recovers the information concerning the —

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temporal orders (high recall), he or she finds it very difficult to "integrate" initial and final temporal successions.

Our proposal concerns the development of temporal inferences when the information is given in a form of a hypothetical situation that the child has to imagine without manipulating anything. As far as the qualitative and quantitative aspect of reasoning is concerned, our basic hypothesis is that the presence of numerical givens make the task resolution easier, because this type of verbal information (quantitative) lends to the construction of algebraic integration rules by the child. We point out that the works has been concerned with the "additive structure" were based on numerical informations without to take into account the informations of temporal orders (Carpenter, Moser & Romberg, 1982). In other studies, it appears that the algebraic integration rule construction is observed in the qualitative tasks, but only with increasing age of the children (Levin, Wilkening & Dembo, 1984).

THE THEORETICAL MODEL

Below we present the theoretical model proposed by Crépault (1978,1981,1983,1988,1989), based on the structural stability of certain temporal subsystems. The basic hypothesis of this model is that there are certain pairs of relations which have priority during the course of cognitive development. The model suggests that temporal relations are initially dyadic and become triadic latter. The model suggests furthermore two main types of knowledge which the subject applies in a given situation: General knowledge and inferred knowledge (this latter type is constructed from the first one). The concordance or discordance among the two types of knowledge leads, through mechanisms that we do not describe here, to cognitive systems structurally stable or unstable, respectively.

The structurally stable or unstable systems are defined by decidable (Rd) or undecidable (Ru) relations, the Rd being the one leading to only one possible valid inference and the Ru leading to many possible valid inferences.

The stable and unstable systems structure is comprised of the follow types:

Stable systems	**Unstable systems**
1. {Ru, Ru, Rd}	1. {Rd, Rd, Ru}
2.{ Rd, Rd, Rd}	2. {Ru, Ru, Ru}

One can consider, for example, three stable cognitive systems, each of which is composed of two subsystems of the type (Ru,Ru,Rd) which, in the domain of temporal relations are:

S-I system: [AF2--AF1--Δt+] and [BF2--BF1--Δt-]

S-II system: [AF2--BF1--Δt+] and [BF2--AF1--Δt-]

S-III system [AF2--AF1--Δt-] and [BF2--BF1--Δt+]

Signification:

AF1: starts after; AF2: finished after; BF1: starts before; BF2: finished before; Δt+: more duration; Δt-: less duration

These systems constitute the structural aspect (pole) of the model. Finally, we also suppose that there is a functional aspect (pole) to the model.That is, when a dyadic information is presented to the subject (e.g. AF1, BF2), it is possible that the two parts of this information activates two subsystems (e.g. the two subsystems of the S-I system).In this case we hypothesize the existence of "primary" rules (R-1) that the subject uses first and "secondary" rules (R-2) that the subject uses afterwards,in order to solve the task.e.g.

a/ R-1 : t2 ; R-2: t1
b/ R-1 : t2 ; R-2: Δt etc,

In spite of the fact that the model work only with decidable and undecidable information concerning inequality and that, in this study, the material chosen use verbal information concerning equality, the material does not lend itself to the application of the model. Nevertheless, we will take this into account when we discuss undecidable verbal information concerning inequality.

METHOD

Material

The material consists of seven decidable type problems (where there is only one correct answer possible) presented by two sentences written on cards. There were two kinds of problems. Subjects were presented either the relative succession orders for beginnings and for endings of two lamps or the relative duration and the succession orders for beginnings or for endings.

The problems were presented in one of two forms: a qualitative form where the relativity is expressed by the terms "before" and "after" and a quantitative form where the relativity is expressed by numbers (e.g."the red lamp goes on at 9 o'clock,the green one at 11 o'clock").Thus,subjects were presented a set of fourteen problems.

Two types of wordings are distinguished:

1)Equality type problems

a) Where the two orders are simultaneous either for the initial point(this will be indicated by t1=) or for the final point(which will be indicated by t2=). These are problems known as 1 and 2.

b)Where the two durations are equal.These are problems number 3 and 4.

2)Inequality type problems

a)Where the two orders are not simultaneous (t1≠ and t2≠).This is problem number 7.b)Where the two durations are unequal (Δt≠) and the two orders are not at all simultaneous(t1≠ or t2≠).These are problems numbers 5 and 6

The problems can be represented as follows:

	1	2	3	4	5	6	7
wording	t1=	BF1	BF1	Δt=	AF1	Δt≠	BF1
	AF2	t2=	Δt=	AF2	Δt≠	BF2	AF1
Inference	Δt?	Δt?	t2?	t1?	t2?	t1?	Δt?

Signification: t1: initial temporal order (order of ignition); t2: final temporal order (order of extinction); Δt: duration (of the lamps);

=: simultaneous order or equal duration

BF (1 or 2): "before" for the beginning (1) or for the ending (2) succession

AF (1 or 2): "after" for the beginning (1) or for the ending (2) succession

Examples of presented sentences

a) *Qualitative form*

-Simultaneous orders (t1=): ".goes on at the same time" or (t2=) "....goes out at the same time"

-Equal durations (Δt=): "...shines for the same amount of time"

-Unequal durations (Δt≠)" the red lamp shines for more time than the green one"

-Non simultaneous orders (BF1)" the red lamps goes on before the green one" or (AF2)" the red lamp goes out after the green one".

b) *Quantitative form*

-Simultaneous orders (t1=): "...goes on at 9 o'clock" or (t2=) "....goes out at 13 o'clock"

-Equal durations (Δt=) "...shines for 3 hours"

-Unequal durations (Δt≠): " the red lamp shines for 2 hours, the green one for 4 hours"

-Non simultaneous orders (BF1): " the red lamp goes on at 9 o'clock, the green one at 11 o'clock" or (AF2) "the red lamp goes out at13 o'clock, the green one at 12 o'clock"

Procedure

The Experimenter (E) familiarized the subject with the task. Then, he E presented the first card and asked the subject to read it aloud. The information remains available to the child. Finally, E asked for a judgment from the child , without insisting on a justification, given the high quantity of problems presented. The qualitative and quantitative forms as well as the equality and inequality types of verbal information were given at random.

Subjects

Fifty seven subjects participated in the experiment,from each of the following three age groups: 19 six-years-old, 20 nine-year-old and 18 twelve-year-old subjects. The gender of the subjects was not taken into account.

RESULTS

We present here the principal results concerning the relative difficulties of the verbal problems according to: (I) the equality/inequality distinction, (II) the qualitative/quantitative form of the verbal information and (III) the proposed theoretical model.

Analysis of the data shows that(TABLE I):

TABLE I: Mean frequency (qualitative and quantitative) of correct answers according to the equality/inequality distinction

problem	Inferred relation	Age		
equality		6 years	9 years	12 years
t1=, t2≠	Δt?	34	82	91
t2=, t1≠	Δt?	15	67	94
Δt=, t1≠	t2?	45	55	70
Δt=, t2≠	t1?	26	33	63
inequality				
t1≠, Δt≠	t2?	55	80	80
t1≠, t2≠	Δt?	45	78	95
Δt≠, t2≠	t1?	45	57	80

a/ The t1= type problems are solved from the age of 9 years. The observed errors for 9 year old children are of the type Δt-.

b/ The t2= type problems are solved at around 9 to 12 years of age. The observed errors are either of the type Δt- or Δt=. Thus, one find a clear difference for 6 and 9 years old children for the rate of the success in inferring duration based on the initial succession orders (t1≠) and in inferring duration based on the final succession orders (t2≠).

c/ The (Δt=) type problems (double temporal decalage) pose serious conceptual difficulties for all the subjects. The observed errors in every age are of the types (t1=) or (t2=).

d/ The inequality type problems are partially solved by the younger subjects, for whom the inferences concerning the final succession orders seem to be slightly easier than the inferences concerning duration or initial succession orders. This is not true for the 12 years old . Furthermore, the observed errors found among the inequality type problems are different than those found among the equality type problems (which normally concerned an equality form).

TABLE II : Mean frequency of correct answers according to the qualitative/quantitative distinction

Problem type	**Condition**	**AGE**		
		6 years	9 years	12 years
EQUALITY				
t1=	C1	42	90	94
	C2	26	75	89
t2=	C1	10	60	94
	C2	21	75	94
at=	C1	40	60	77
	C2	31	30	55
INEQUALITY				
t1=	C1	68	85	89
	C2	42	75	72
t2=	C1	58	70	100
	C2	31	85	89
at=	C1	37	60	83
	C2	53	55	78

C-1 : Quantitative condition

C-2 : Qualitative condition

The data analysis shows :(TABLE II):

a) Generally, the problems are solved better when using the quantitative form than in the qualitative one. This concerns every type of problem and every age, with some exceptions concerning younger subjects especially.

b) The evolution of reasoning competence is very important, given the difference of correct answers according to age, for both types of reasoning conditions(qualitative and quantitative).

c)For younger subjects (aged 6 to 9 years), the effect of the reasoning condition appears to reverse itself for the t2= type problems; success was slightly better for qualitative reasoning condition for this type of problems.

As far as the presented theoretical model is concerned,we have obtained the follows results (TABLE III):

TABLE III : Use of functional rules according to the qualitative/quantitative distinction for the inequality type problems
(grouped according to responses despite of different rules)

	Structural system	Functional rules	Qualitative reasoning	Quantitative reasoning
Group 1	S-II S-I S-I	R-1 :AF R-1 :t2 R-2:t1 R-1 :t2 R-2:Δt	20	22
Group 2	S-I S-II	R-1 : AF R-1 :t2 R-2: Δt	8	12
Group 3	S-III	R-1 : AF	3	5
Group 4	S-III S-I S-III	R-1 :AF R-1 :AF R-1 :t2 R-2:t1	2	4
DIVERSES			24	14

a) The principal functional rules are those of group 1.They are the rules leading to the correct answer patterns.

b) The grouping of different systems does not allow for an analysis of the genetic evolution according to these systems (precocity of the acquisition of one system in relation to the others).

DISCUSSION

The principal experimental results show a considerable effect for the facility of reasoning based on the numerical givens (quantitative reasoning). Thus, it appears that the numbers make the integration and the processing of the information easier, because of its concrete character. The problem thus becomes "calculable". This facilitative effect concerns, in general,older children aged 9 to 12 years. Furthermore,we have observed an "reverse" effect for this facility in favor of qualitative reasoning for young children aged 6 to 9 years, concerning the t2= type problems (where the inference concerning duration is based on the initial succession orders, t1≠).

This "paradoxical" effect is perhaps comprehensible on account of two facts. On the one hand,the type t2= problems are the most important cause of difficulty for the young child.On the other hand,the child is not capable at an early age, to construct algebraic rules because his cognitive functionement is characterized especially by sequential decision rules which latter become algebraic integration rules (Levin, Wilkening & Dembo,1984; Kerkman & Wright,1988; Wilkening,1988).

An interesting question to point out concerns the results of studies on "additive structures" where subjects have to find out what the final state is, knowing the initial state and the transformation.These final-state problems are much easier than others (Vergnaud,1982). Are there any connections between these results and the fact that children found it easier to "judge" the final order of successions knowing the initial order and the duration?

This research raises an important question concerning the decalage between the precocious ability of a young child in duration judgments in experimental situations that uses a set of manipulated objects(Levin and al.,1984) and the relatively low scores of 12 year old children in hypothetical verbal problems concerning similar experimental situations.What is the role of having actual objects as a support and how do the subjects imagine the symbolic information?

The presented model is a first step at grasping,for a wide class of situations, some aspects of the genesis of temporal reasoning.On the one hand this model does not work with the equality type problems .On the other hand ,it does not differentiate always between the response patterns in accordance with the structural systems. Thus, in this study the model has demonstrated its limits and does not permit us to come to strong conclusions concerning its validity.

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THE MULTIDIMENSIONAL NATURE OF THE PRE–CONCEPTS OF NUMBER

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Thirty case studies had been made on the kindergartners' knowledge of the pre-concepts of number which Herscovics & Bergeron have identified as plurality and position. Based on percentages of success to the 20 tasks used, they found a gradation based on the rate of success for the various tasks. This paper investigates their results somewhat further and tries to establish the nature of the dimensionality of the data with a variety of statistical and psychometric analyses. On the basis of these analyses, there seems to be evidence that plurality and position possess distinct underlying factorial structures.

INTRODUCTION

In our investigation of the kindergartners' knowledge of number, we started with an epistemological analysis of the number concept, that is, an analysis of likely patterns of construction by the learner. This was performed using the two-tiered model of understanding proposed by Herscovics & Bergeron, J.C. (1988a) in which the first tier describes the understanding of the preliminary physical concepts, and the second tier, that of the emerging mathematical concept. The preliminary concepts identified were the notion of **plurality** of a discrete set and the notion of **position** of an element in an ordered set (Bergeron & Herscovics, 1988, 1989; Herscovics & Bergeron, 1988b, 1989). According to this analysis, the concept of number, which is viewed as both the measure of plurality and the measure of position, emerges from these pre-concepts. The present paper looks at the data obtained from 30 case studies on the understanding of these two pre-concepts in the perspective of statistical analyses, one of the purposes being to determine if the hierarchies observed are indicative of an underlying cumulative unidimensional scale.

According to the model used, the two pre-concepts of number, plurality and position, are constructed by the children in three consecutive stages corresponding to three levels of understanding named "intuitive understanding", "logico-physical procedural understanding" and "logico-physical abstraction", the logico-physical expression referring to judgements, or actions, exerted on concrete sets of objects

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and transformations thereof. To evaluate the children's pre-numerical knowledge, about 20 tasks were developed in order to investigate the knowledge related to each one of the three levels on understanding for plurality and for position. Table 1 presents these three levels of understanding for plurality and position, along with the percentage of success (N=30) for the different tasks used.

Table 1–Rates of success for plurality (Q) and position (R)

	Intuitive underst		Procedural underst		Abstraction	
PLURALITY - Q	Visual estimation		Generation of a set		Invariance wrt	
	more	100	as much	100	rotation	73
	less	100	more	97	displacement	70
	many	100	less	93	contraction	63
	little	97	one more	77	elongation	60
	as much	63			dispersion	57
					Piagetian test	40
					hidden (bag)	13
					hidden (card-1 set)	7
					hidden (card-2 sets)	3
POSITION - R	Visual estimation		Generation of a set			
	before, ahead	100	before, ahead	100	"renversibilité"	100
	after, behind	100	same time, together	100	variability	80
	same time together	100	after, behind	97	invariance wrt	
	first	100			elongation	73
	last	100			first hidden	53
	between	93			middle hidden	50
					cons. position	33
					cons. ordity	13

Regarding the **intuitive understanding of plurality**, the children had to compare two sets of identical objects on the basis of visual estimation to find out which one had **more, less, many, little** (or few), **as much.** For **position**, again on the basis of pure visualization, they had to decide if an object was **before, after,** if two objects were **together**; in a row, which one was the **first, the last,** was **between** two given objects in a row.

The level of **procedural understanding** of plurality and position, was based on non-numerical logico-physical procedures. These procedures used one-to-one correspondences to generate sets subjected to constraints listed in the second column of the above table.

For **abstraction** in the logico-physical sense, we used as criterion the children's ability to perceive the invariance of plurality and position under various figural transformations. These involved tasks in which sets of objects laid out randomly , were subjected to **rotation, displacement** (keeping the same area), **contraction, , dispersion,** the **elongation** of a single row, or in presence of another one that is kept fixed (**Piagetian test**), **hiding** part of a set by placing it in a partly opaque plastic **bag**, or by a **cardboard** (once with **one single set**, and then, comparing it with **another set**).

In the study of position, one task investigated if the children perceived the **variability** of the rank of an object (removing the first object in a row changes the rank of the remaining ones). The tasks on the invariance of position included the **elongation** of a row of toy horses, and the partial **hiding** of a row of trucks under a tunnel. Another task investigated if the children perceived the invariance of position when one of two parallel rows was **translated** (conservation of position), and if they could still perceive the numerical rank of a designated car in the translated row (conservation of ordity). Finally, we wanted to know if they could imagine that executing the inverse transformation would result in the original state, which is "**renversibilité**" in Piagetian terms.

Looking at the percentage of success, for each of the six cells, we observe a gradation based on the rate of success for the various tasks. The most difficult task for plurality was the one where part of a set is out of sight, leading the majority of children to believe that there are less in the bag than before (7% and 3%). In the case of position, the most difficult task is the conservation of position (two corresponding cars in two parallel rows have the same position but, if one row is moved slightly ahead of the other, they won't reach the ferry boat together, even when they are called in pairs).

But can we infer, from an inspection of the results, that a cumulative unidimensional underlying scale exists? In order to find an answer to this question, a Guttman scalogram analysis was performed.

ANSWER PATTERNS

The Guttman scalogram analysis assesses to what extent individuals' responses can be predicted or "reproduced" from the corresponding scale scores. This approach rests on a conception of dimensionality which is cumulative in nature. "it is a deterministic model of scaling; each value is a single-valued function of the underlying continuum" (Guttman, 1944). The degree to which individual scale scores predict the subjects' response patterns is measured by the coefficient of reproducibility (REP). A value of 0.9 or more is considered as indicative of a valid scale. However, this coefficient is not a sufficient test of scalability because it depends on the proportion of "correct" responses or item difficulty. The coefficient of scalability (PPR) proposed by Jackson (1949) is free from this effect and provides a scalability criterion (the critical value is usually set at 0.60).

In order to analyze the entire set of data using SPSS software (an updated version of the Goodenough-Edward method), the original pool of items had to be regrouped into a maximum of twelve variables. The first five variables are related to the concept of plurality (name ending by Q, for "quantity") and the last seven ones to the concept of position (name ending by R, for "rank"). "COMP" stands for "compréhension" (understanding), "INT" for "intuitive", "PRO" for "procedural", " ABST" for "abstraction", "SAC" for "sac" (bag), "QUAN" for "quantity", "VAR" for "variability", "INV" for "invariance", "TUN" for "tunnel", "CON" for "conservation" "ORD" for "ordity" (predicting the result of enumeration to find the rank).

Plurality

COMPINTQ	– intuitive understanding of plurality
COMPPROQ	– procedural understanding of plurality
ABSTCARQ	– Abstraction: invariance of plurality with respect to the visibility of the objects when some are hidden under a cardboard.
ABSTSACQ	– abstraction: invariance of plurality with respect to the visibility of the objects when hiding them in a bag
ABSQUANQ	– abstraction : invariance of plurality with respect to various configurations of objects (dispersion, contraction, displacement, rotation, Piagetian test)

Position

COMPINTR	– intuitive understanding of position
ABSTVARR	– variability of position
COMPPROR	– procedural understanding of position
ABSTINVR	– abstraction: invariance of position with respect to elongation of a row of objects
ABSTTUNR	– abstraction: invariance of position with respect to the visibility of the objects when some are hidden in a tunnel
ABSTCONR	– Abstraction: conservation of position wrt a translation
ABSTORDR	– abstraction: conservation of ordity wrt a translation

Three different scales were defined: two separate scales specific to plurality (INTUIQ) and position (INTUIR), and a unique scale combining the variables related to the understanding of both plurality and position (INTUIQR). Furthermore, two separate sets of division points defining the scale scores according to two different rules were used, providing us with two sets of analyses, Set 1 and Set 2. Set 1 allows a maximum of one error when the number of items per variable is three or more. Set 2 does not allow any error. Both sets are conservative, the second one more so than the first one. Based on the very high rate of success of items related to intuitive and procedural understanding, shorter versions of INTUIQ (7 items), INTUIR (4 items) and INTUIQR (11 items) were used.

Table 2 — Summary of Guttman Scale Analysis

	INTUIQ (Plurality)		INTUIR (Position)		INTUIQR (Plurality & Position)	
	Set 1	Set 2	Set 1	Set 2	Set 1	Set 2
REP	1.00	.91 (.89*)	.85	.85 (.78)	.89	.86 (.82)
PPR	1.00	.58 (.66)	.24	.22 (.13)	.35	.30 (.39)
Yule's Q (% neg.)	0	.20 (.05)	.10	.38 (.50)	.08	.23 (.20)

* The results of the short versions are between brackets

The results, summarized in Table 2 (under INTUIQ) show that only the understanding of plurality as measured in this test and for this group of subjects seems to form a cumulative unidimensional scale (PPR percentage close to or larger than 0.60). Thus, the gradation observed for plurality in Table 1 (based on success rates) can now be interpreted in terms of an underlying unique cumulative dimension according to the scalogram rationale.Based on increased difficulty, the following variables can be ordered within the underlying unique dimension of the concept of plurality: intuitive understanding, procedural understanding, abstraction (change of configuration), abstraction (visibility of objects, bag), abstraction (visibility of objects, cardboard).

Furthermore, within the framework of this method, no unique scale seems to underly the concept of position or, a fortiori, the combined concepts of plurality and position when considered jointly. This raises the possibility of multidimensionality.In order to further investigate the multidimensionality according to different rationales, namely, explaining variance, correlations and reliability, a series of factor analytic type of analyses — principal component, exploratory and confirmatory factor analyses, as well as correspondence analysis— were performed. A tentative trial was made to identify possible underlying factors, based on these methods and provided some interesting results. However, our initial sample is too small to warrant generalizations.

ANALYSES OF A FACTOR ANALYTIC NATURE

Principal component (PC) analysis can be conceived as a transformation of the underlying measurement axes in order to maximize variance. Because of this emphasis on variance, the 14 items of the test with null variance — namely, the items related to the intuitive and procedural variables were excluded for they did not contribute to the analysis. However, since the remaining items essentially cover the same abstraction variables, the PC results should complement our understanding of the dimensionality. More precisely, the remaining items cover all the ABST-variables defined earlier. As previously, three separate analyses were performed, one for the concepts of plurality, one for the concept of position, and a third one combining both concepts. Table 3 presents the results of these analyses.

In Table 3, the joint analysis (**Q&R**) shows the existence of six orthogonal factors (uncorrelated) explaining 76.4% of the variance. Three of those factors pertain to the understanding of plurality (cf. analysis of Q) namely, the same abstraction variables that are found at the most difficult part of the cumulative Guttman scale: abstraction — change of configuration, abstraction — visibility of objects (bag), abstraction — visibility of objects (cardboard). This seeming contradiction (a unique Guttman dimension vs three principal components or factors) is resolved if we note that

> *multidimensional tests can be confused as unidimensional tests if the multiple dimensions have proportional contributions to each item. In such a case, scores on a test would represent a weighted composite of the many underlying dimensions (Hattie, 1985).*

The other three factors are related to the concept of position : conservation of position, conservation of ordity and a mixed factor defined by three variables: variability of position, invariance of position with respect to elongation and visibility of objects. This last factor presents a negative loading of −0.65 for the variability of position, as opposed to two large positive loadings for the visibility of objects (0.93 and 0.88). Furthermore, the Ø correlations between the underlying items of the variability of position (ABSTVARR) and the visibility of objects (ABSTTUNR) are negative (−0.47 and −0.33). This suggests possible separate factors for those two variables.

Table 3 – Summary of the principal component analysis

Analysis	%var. explained	# factors	Description of factors	% var. explained
1. Q Plurality	72.5	3	1 — ABSTCARQ	37.7
			2 — ABSTSACQ	24.8
			3 — ABSTQUANQ	10.0
2. R Position	72.4	3	1 — ABSTORDR	30.9
			2 — ABSTTUNR ABSTVARR ABSTINVP	25.1
			3 — ABSTCONR	16.5
3. Q & R Plurality and Position	76.4	6	1 — ABSTQUANQ	26.5
			2 — ABSTVARR ABSTINVR ABSTTUNR	14.7
			3 — ABSTSACQ	11.9
			4 — ABSTCARQ	11.4
			5 — ABSTCONR	6.3
			6 — ABSTORDR	5.5

When the focus of the analysis shifts from explaining the total variance to explaining the matrix of intercorrelations, a factor analysis (FA) model is used: it allows for orthogonal and oblique (correlated) factors. The resulting joint analysis of plurality and position yields the same factorial structure as for PC, with an explained variance of the same order of magnitude, namely, 69.4%. Most factors are orthogonal (with the exception of factors 1 and 6 presenting a low correlation of approximately 0.30). A confirmatory maximum likelihood analysis (variance explained: 63.8%; probability of chi-square test = 0.57) confirms those results and further supports the seeming distinct factorial structure of plurality and position. Finally, graphical displays of dual-scaling analyses (a version of correspondence analysis) leads to the same conclusion.

For all the methods, it is important to note that the joint analysis combining both plurality and position, recovered the same structures as the separate analyses and no new mixed factor appeared. The consistant absence of confounding of the factors pertaining each to position and to plurality, lends empirical support to the conceptual distinction proposed between these two concepts.

CONCLUSION

We have seen that the empirical dimensions of the data lends support to the theoretical definitions of the variables. The methods we used, the Guttman scalogram and a variety of factor analytic models and techniques, support the conclusion that the set of items seem to be multidimensional in nature, but with distinct structures for the two preliminary physical concepts of number –plurality and position. Furthermore, the factors underlying plurality seem to constitute a cumulative unidimensional scale.

Whether the intuitive and procedural understanding of these two pre-concepts, which were not included in the factor analytic analyses, define four uncorrelated dimensions of the same type as the six ones pertaining to the abstraction of plurality and position is a likely proposition but cannot be answered at this stage, since the success rate of the underlying items was very high and therefore, caused the correlation matrix to be ill-conditioned. It is worth noting that the structure found proved to be quite stable and this, over the various methods used.

A final caveat is in order with respect to the limitation of the study stemming from our relatively small sample of subjects (30). Further analyses with larger samples are envisaged in order to confirm the proposed multidimensional structure of the pre-concepts of number. It should however be noted that the group studied was highly homogeneous with respect to age (mean: 5 years and 9 months; standard deviation: 3.5 months) thus eliminating variability of age as a possible source of error.

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PROTOCOLS OF ACTIONS AS A COGNITIVE TOOL FOR KNOWLEDGE CONSTRUCTION

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Abstract: *This paper endeavours to develop a model for the constructive activity (of a learner) which potentially can result in the cognitive development of (certain types of) mathematical knowledge by the learner. This model will be a special case or application of the general and fundamental theses: knowledge is the (cognitive) product of material and/or mental actions (operations) and the reflection by the learner on his/her actions, their structural properties and their results or outcomes. What are specifically the actions, their means and tools, their outcomes (products, results), and how they are reflected upon depends on the intended knowledge and partly also on the learner and his/her cognitive status.*

Introduction

The guiding context and the main stimulating basis for this paper is the genetic epistemology as developed by J. Piaget, see among many possible references PIAGET (1967, 1970 and 1972) and BETH/PIAGET (1966). Especially appropriate for my intentions are interpretations of Piaget's positions given by E. VON GLASERSFELD (1988). He cites Piaget among others with the following sentences which could be taken as a general framework for what follows:

> "Toute connaissance est liée à une action et connaitre un objet ou un événement, c'est les utiliser en les assimilant à des schèmes d'action ... Connaitre un objet implique son incorporation en des schèmes d'action, et cela est vrai des conduites sensori- motrices élémentaires jusqu'aux operations logico-mathématiques superieures." (Piaget 1967, 14–15, 17).
>
> "Dans le cas de l'abstraction logico-mathématique, ... ce qui est donné est un ensemble d'actions ou d'opérations préalables du sujet lui-même, avec leurs résultats." (Piaget 1967, 366).
>
> "... la construction logico-mathématique n'est à proprement parler ni invention ni découverte: procédant par abstractions réfléchissantes elle est une construction proprement dite, c'est-à-dire produisant des combinaisons nouvelles." (Loc. cit. 367).

In a way my paper tries to substantiate notions like "construction", "reflection" and "abstraction" by offering an operational model for the case of mathematical concepts.

What does it mean to construct one's knowledge?

One central thesis of constructivism asserts that all kinds of knowledge which an individual possesses is his or her personal construction. This refers as well to the knowledge which is acquired in a didactical situation like it is provided in a traditional school setting. In a rather negative way this thesis can be interpreted such that knowledge cannot be received in a passive and only receptive way by the learner from any kind of teacher (human, book, computer). But this does not tell us anything about how we could conceive of such a personal construction of knowledge. The term "personal construction" is a very suggestive one which might seduce one not to ask for an explanation and a more detailed description of the (cognitive) processes which make up the knowledge construction and which lead to certain knowledge products. Assertions like "knowledge is the product of the activity of the learner" are not of much help for didactical purposes like the planning of teaching-learning situations. What is needed is a deeper insight and understanding of how knowledge is developed by the learner through his/her activity and of what are the mutual relationships and dependencies between the activity and the resulting knowledge. We should be able to describe in a much more operational way what "constructing one's knowledge" can mean and how one could prompt the learner by didactical measures to carry out the adequate constructive activity.

Alienation of mathematical knowledge

Before treating the special topic of this paper some more general remarks are appropriate. One widespread characteristic of mathematical knowledge as it is acquired by students at school and even at universities is what I would call its separatedness from the knowing individual. Based too often on rote learning the knowledge remains alienated and is not effectively incorporated and integrated into the cognitive system of the student. Many students consider (school) mathematics as being irrelevant for their life in a very deep and not just utilitarian sense. They rarely have experienced mathematical concepts and methods as tools and means for structuring, organizing or planning their personal activity beyond solving assignments in school. Mathematics for many students never gets their own activity, it remains something which others have done (who really know how to do it) and devised and which can only be imitated (for instance by the help of automized algorithmic routines at which the student works more like a machine than like a conscious human being). In other words, mathematics mostly is not part of the personal experience and the reason for this very likely is that the mathematical knowledge of the students was not (or only in an insufficient way) the result of structuring and organizing their own experience. This is the more remarkable when one considers the widespread consensus that mathematics is primarily a human activity and is essentially constituted and created by human activities. This states the apparent paradox that something which has been created by humans through their activity is experienced by so many people as something completely external to and opposed to the individual. This feeling is strongly related to teaching methods which are based on the "outside-to-inside" approach to learning and knowledge acquisition. To overcome this rather detrimental situation didactical means have to be employed which offer to the student frequent opportunity to realize mathematics as part of his/her own activity and to experience mathematical knowledge as the outcome of this personal activity. In this way mathematics could become the mental property of the student which then can be a cognitive means for guiding and organizing appropriate fields of activity and experience (including perception). Essentially, this conception reflects the strictly constructivist position that one can only genuinely understand what one has (re)produced oneself. This of course does not mean the invention of mathematics by the students but the active reconstruction by and through one's own actions and reflections (which must be guided and controlled by some kind of teacher).

Modeling the construction of mathematical knowledge

We now turn to the more specific model for knowledge construction in the field of mathematics. This model should be taken firstly as an epistemological one as by it objective-general features of mathematical concepts and their genesis are to be reconstructed and modelled. But it can be taken as a psychological model as well in so far as it intends to model cognitive processes which potentially lead to the (re)construction of the respective mathematical content by the learning individual. Thereby I base my assertions on the thesis that effective models for cognitive processes have to take into account the outcomes of pertinent epistemological analysis. In other words: the (re)constructive activity carried out by the learner has to be organized along the epistemological structure of the knowledge to be constructed. This epistemological structure will – sometimes in a mediated way – reflect features and determinants of the original constructive processes which constituted the content of that knowledge. Finally it should be remarked that the proposed model is not intended to be universal. Its application and applicability depend on the specific subject matter and have to be based on a thorough analysis of it. We will start with a general description of the model which will be followed by several examples.

The model is based on the epistemological analysis which states that many mathematical concepts are structurally and genetically related to certain actions and their products cf. for that DÖRFLER (1984) and DÖRFLER (1988). The specific form of this relationship of course depends on the respective concept. The (cognitive) reconstruction of such a concept starts (in the model) from carrying out the pertinent actions whereby these can be actual actions with material, concrete objects or mental operations with imagined (mental) objects. The (actual or imagined) reproduction of actions of other people which one has observed can play a similar role. The central question is by which means the individual mentally (re)constructs the mathematical concepts based on the carrying out of the pertinent actions. Clearly the concept is not just the action nor plainly its schema or structure though those form an integrating part of the concept.

As already pointed out above, reflection on the actions is the way and means which lead to the mental constitution of the adequate cognitive structures which then represent the content and structure of the intended concept. As a main tool for the reflection of and the reflecting upon the action by the model is proposed what I want to call a *protocol of the action*. Such a protocol by the use of perceiveable objects (like written signs) tries to note and to describe the characteristic and relevant stages, steps and outcomes of the actions.

The completed protocol will be a static structure of signs forming a certain pattern which represents certain relationships between the signs. In a way, this pattern can be treated as the resulting outcome of the actions or rather as a (symbolic) representation of it. But, like good minutes of a meeting not only state the decisions made but also give an account of the discussions and deliberations leading to them, the action protocols must allow to trace the stages of the process which lead to the final protocol. Thus the protocol must reflect in some way a certain temporal pattern which is fundamental for the action and the constituted mathematical concept as well.

It might be appropriate here to emphasize that the related mathematical concept in most cases reflects only very general and schematic aspects of the actions onto which the attention of the learner must be deliberately and consciously be focused (for instance by cues given by the teacher). To "see" mathematical features in one's own actions is by no means trivial and further depends on the interests and the motivation of the learner. The problems related to this can not be treated here as though they are very important for the didactical realization of the presented model.

In many cases the (or a) mathematical structure which can be associated with actions of a certain type consists of transformations exerted on the objects on which the actions are carried out, or of relationships which are induced by the actions between those objects. The protocol of the action, i.e. the cognitive reconstruction of the concept, should reflect or permit to reconstruct these transformations and/or relationships. In this sense the signs for the protocol play the role of objects on which schematized actions can be carried out which induce just the constitutive transformations and/or relationships. In other words, the protocol may be used to carry out by and on it just those steps of the actions which are the "germ" of the intended concept (and which have been recorded by the protocol): The protocol records or represents the (mathematically) essential features of the actions and permits to replicate them.

The intended reflection upon the actions consists in the recording of the protocol which gives an account of the actions and their main goal. Since the protocol is a permanent record (in contrast to the passed-by action) it can be used over and over for analyzing the action, for instance by replicating it in a schematized way on the protocol. For the individual cognitive process it will be very important that the protocol is a record of (the mathematical characteristics of) one's own actions. The protocol should be a personal account of what one is interested in when carrying out the action. If the protocol is not experienced as an account of one's own actions it cannot be really understood and will not be related to the actions in a meaningful way.

In a didactical context this means that the learner has to be guided by the teacher to carry out the actions, to find out those objects and stages which are of essential relevance for the action and to produce a protocol which represents these objects and stages. It should be emphasized that an action protocol is not uniquely determined by the actions.

It will depend on how the learner structures and organizes his actions and further it will depend on which means are available for obtaining a protocol. A protocol can be formulated by different media and in different sign systems. Possibly one will start with a verbal protocol which by using natural language gives an account of what are viewed to be the most important aspects in the flow of the action (from a mathematical point of view). This protocol can sometimes be translated into a protocol using geometric or iconic objects and in other cases it can be expressed in an algebraic language. It might even be useful to produce several distinct protocols and to interpret the actions in all of them thereby establishing a kind of translation from one protocol to another. One should never forget that the protocol receives it meaning and the rules for how to interpret it and for how to act on it from the actions which it represents.

Since in most cases the protocol will be a schematic representation of certain aspects or features of the actions and their objects it is potentially much more general than the action itself. This will often be reflected by that the signs used in the protocol have the character of variables or can at least be considered to be variables. Thereby the same protocol can be used as an account of many different actions as long as these have the same "mathematical structure" as expressed by the protocol and represented by the respective mathematical concept. As a didactical consequence of this one should have the learner experience this varied use of the protocol in as many different situations as possible. Thereby the protocol might be used also as a tool for structuring and planning concrete actions by substantiating the general schema (the protocol) in a given concrete context. For this it is important to keep in mind that the protocol by its very form and structure permits to carry out certain specific steps of the actions or at least to reproduce them. These are those steps which are constitutive for the related mathematical concept. This acting on the protocol in a paradigmatic and prototypical way exhibits which from a certain point of view are the essential stages in the actions and which relationships are the important ones.

The protocols usually are patterns of material signs and objects and as such can become the topic of further analysis and study. This is what more or less can be termed as a mathematical analysis and by it various properties of the protocols, possible transformations of and relations among protocols etc. can be devised and investigated. Most important are actions which use the established protocols as objects to be acted upon. This can give rise to a new layer of concepts (and protocols) thus establishing a kind of hierarchy of concepts and protocols. But I will not pursue this topic here any further. As another remark I refer to the habit to introduce names for the protocols. These names can be words of the natural language or any other conventional signs. As one can see from the examples this naming of protocols permits to establish compound protocols of more complex actions from simpler building blocks.

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Examples for and applications of the model

In the following I will list several mathematical concepts together with a sketch of how the model of the action protocols could be applied to them. These sketchy remarks are not meant as didactical programs but those could and should be developed starting from them.

Natural numbers. The action clearly is any kind of counting activity. Protocols may be the verbal uttering of the number words; using fingers or other representatives of the counted objects; lists of strokes. All these give a precise account of the main steps of the counting procedure with regard to the question "How many?". On the other hand those protocols permit the replication of these essential steps on themselves – the protocols are prototypes of (sets of) countable objects.

Decimal numbers. Again counting activity is the starting point but now it is organized in a certain way by using counting symbols of the values 1,10,100,1000 and so on and by always changing to the next higher unit when possible. As in all the examples we (have to) assume that the learner is in a cognitive stage which enables him to carry out adequately the actions. For instance here he must be able to form mentally higher order units which again can be counted. That granted, the resulting protocol will be something equivalent to the decimal representation of the number of counted elements. Clearly this representation reflects the main phases or steps of the "decimal counting" and in a very condensed form permits its replication (when assuming that all the higher order units have already been formed). A similar approach holds for other bases of the number system.

Decimal fractions. Consider a measuring process by which one wants to exhaust a quantity (say, less than one unit) by tenths, hundredths, thousandths and so forth of the unit. This process naturally leads to a decimal fraction as its protocol and result. By reading this protocol one can mentally reconstruct the process: first take d_1 tenths, then d_2 hundredths and so on until the whole quantity is obtained. Here like in other cases the protocol can further be used as an action plan for producing an object (a quantity) which just will give rise to the protocol when the actions are carried out on it. Again one can use different bases for the measuring process.

Fraction. A fraction can be viewed as a very curtailed or abridged protocol of a measuring process: m/n represents "m copies of quantity A are equal to n copies of quantity B" or "m copies of the nth part of quantity A give the quantity B". Again there is a wealth of different activities possible around this relation of m/n to measuring processes from which for instance concepts like equivalence of fractions, ratio of quantities, addition of fractions could be derived. This example makes very clear that a pattern of signs has to be explicitly related to the actions which constitute (part of) the meaning of the signs (as an action protocol). In this specific case the protocol m/n itself does not permit replication of the action but this gets possible if m and n are replaced by protocols (for the related counting processes). This corresponds to combining the counting and measuring processes. The number symbols can be viewed as names for the respective protocols (or for the actions themselves) and m/n then gets a composed protocol. Still, this example makes clear that producing and reading a protocol heavily makes use of signs whose meaning is determined by convention.

Matrix. Consider two sets A, B of elements (objects, properties etc.) and some kind of measuring the "value" of a relation between elements of A and of B. A protocol will very likely take the form of a matrix. A matrix such can be viewed as a way of noting the values of (binary) relations and the arrangement of its elements in lines and columns mirrors the stepwise process of determining these values. In this sense then a matrix is not just a static array of elements but it has inherently a dynamic structure or even several ones (e.g. scanning by lines or by columns); it is a schema of writing down symbols. Conversely the idea of the matrix as the intended protocol determines the order of the steps in any measurement of this kind. In a very similar way any kind of table can be viewed as the protocol of some measuring (in a broad sense).

Graph. The drawing of a graph can in a way very similar to the previous example be considered as the recording of a relation in a given set of elements. "Reading" this protocol gives a full account of the (formal) properties of the respective relation, a weighted graph even reports on the strength of the relation between any two elements. As is well known this type of protocol has become the topic of extensive mathematical investigations (Graph Theory).

Angle. Angles are mostly viewed as static geometric objects which are defined in one way or another. But they can also be used as the protocol of a rotation (around the vertex of the angle): the first side is rotated into the direction of the second. The very same protocol then results from angles which usually have to be considered as equivalent. These equivalent (static, figurative) angles correspond to actions which very naturally everybody will view to be the same ones. Here again the action can be replicated directly on the protocol or its replication can easily be imagined.

Algebraic terms. These can be viewed as protocols recording some calculations or better the schema of the calculations. Conversely, a term tells me which calculations are to be made in which order (according to conventions agreed upon). Tree representations can play an equivalent role. In my opinion it would be very helpful for an adequate understanding of elementary algebra if the concept of algebraic terms and their manipulations were based on such an understanding: Terms as schematic protocols of (my own) calculations which I can use to give an account of them to myself or to others. Like in all previous examples a term in this way permits to reflect on the calculation (e.g. to study the influence of certain numbers on the outcome).

Concluding remarks

These examples should make clear that the proposed model for the cognitive construction is intended to at least complement the prevailing static and figurative conception of mathematical concepts (at schools) by bringing into the foreground constitutive actions. These are in my opinion an essential part of the meaning of the concepts and if the latter are acquired via the actions and their protocols there is a good chance that the concepts will be used as mental tools for thinking, for mental operating and for the planning of concrete actions. So the question for a definition of a concept should not (only) be "What is xy?" but (also) "Which actions can be recorded and/or organized by using xy?"

The list of examples could be extended further by treating concepts like: graph of a function, function, vector, geometric figures (via their constructions, or LOGO programs where appropriate). These still can be related to rather concrete actions whereas other concepts (can) give an account of mental operations. This applies to all infinitesimal operations: sequences, series, limits, derivatives, integral etc. But again I consider it to be worthwile and important to emphasize the dynamic and operational aspects of these concepts and the possible protocols of the respective processes.

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REPRESENTATIONS DU FONCTIONNEMENT D'UNE PROCEDURE RECURSIVE EN LOGO

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(Conference will be in english)

__*Summary:*__ *This research concerns 14-15 years old students in a classroom , having a LOGO experience prior to the study . We present different teaching situations with a constructivist approach about recursion separating difficulties pointed out in a preexperimentation . Two steps are necessary to be able to write a recursive procedure : to understand the internal processes of a recursive procedure and to appropriate a static representation of the procedure. The study is focused here on the analysis of the representations that students have about the operating system for a recursive procedure .*

I) Analyse a priori

L'écriture de procédures récursives nécessite le passage d'une représentation **dynamique** liée à **l'exécution** , en termes d'actions , à une représentation **statique** de la procédure , en termes d'états (J.Rogalski, G. Vergnaud , 1987) . Cette représentation statique ne peut apparaitre qu' après une connaissance du **dispositif informatique** suffisante pour se créer un modèle de fonctionnement : il faut se représenter les problèmes sous une forme **compatible** avec le **dispositif informatique** (J.M Hoc,1987). L'écart , qui existe entre la compréhension du fonctionnement du dispositif informatique et l'écriture de procédures récursives, est lié au fait que l'écriture **récursive** nécessite la mise en place d'un **schéma fonctionnel statique** du type : pour construire l'objet de niveau **n** , on suppose construit l'objet de niveau n - 1 .

Nous avons voulu séparer les différents types de difficultés mises en évidence dans la préexpérimentation (C.Dupuis , M.-A. Egret , D.Guin , 1985) : l'emboitement de procédures, l'exécution de procédures récursives fournies et l'écriture de procédures récursives.

L'**emboitement** de procédures différentes est le prolongement "naturel" des connaissances acquises en programmation structurée. Le début du travail est donc consacré à la recherche de l'**invariance d'emboitement** . Celle-ci évite le problème de la **suspension de l'exécution** (comme dans une procédure récursive) tout en préparant à la **représentation statique** du programme, nécessaire à l'écriture de procédures récursives : c'est un premier pas vers l'élaboration d'un schéma fonctionnel statique. Cette phase nous a permis d'analyser les difficultés propres à l'emboîtement, notamment les difficultés de codage et les modalités de gestion des relations (C. Dupuis, D. Guin, 1989) .

L'**exécution** de procédures récursives fournies à l'élève lui permet de se créer une représentation du **fonctionnement du dispositif informatique** , à condition qu'on lui **propose des modèles** de fonctionnement de la récursivité lorsqu'il en ressent le besoin. L'**auto-référence** non terminale apparait comme un cercle vicieux dans un premier temps. Il faut convaincre l'élève que l'ordinateur accepte ce type de programme, c'est-à-dire qu'il peut l'exécuter. La conception **spontanée** de l'auto-référence (que l'on voit apparaître dès que son existence est admise) est une forme de **retour au début** du programme, et s'apparente au schéma spontané de la répétition : faire une action et recommencer. Or le schéma spontané de la répétition s'accommode mal d'un des aspects essentiels de la récursivité : la suspension de l'exécution de la procédure appelante pour attendre la fin de la procédure appellée (A. Rouchier, 1987).

Nous aurions pu intervertir les séances 1 (invariance d'emboitement) et 2 (prévisions et exécutions de procédures récursives fournies), cela aurait sans doute favorisé le passage d'une représentation dynamique à une représentation statique , mais la séance sur les courbes était pour nous un moyen de motiver de jeunes élèves à une étude de la récursivité . Notre démarche d'enseignement de la récursivité est une voie intermédiaire entre celles proposées par P. Mendelsohn et A. Rouchier (P. Mendelsohn , 1988).

2) *Présentation des situations d'enseignement*

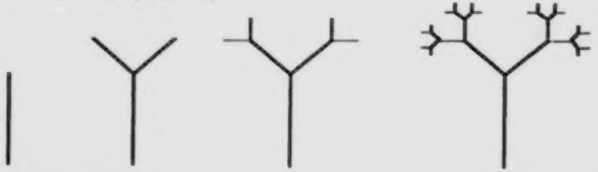
Les situations présentées ici correspondent à une vingtaine d'heures d'enseignement.

Séance 1 : **Les courbes** (Recherche de l'invariance d'emboitement)

……

Voici trois familles de courbes . Sur une même ligne sont dessinées quatre courbes de la même famille . Choisissez une famille , puis écrivez les programmes correspondants à chaque courbe , **en utilisant à chaque fois le programme précédent** .

Une des familles de courbes (les arbres) :



……

__Analyse de la situation :__

L'activité est complexe et comporte des difficultés de structuration et de coordination, points essentiels dans la première phase de notre expérimentation (C. Dupuis , M.A. Egret , D. Guin , 1988) mais ces courbes plaisent esthétiquement aux élèves , ils ont d'autant plus d'ardeur au travail ! Ces familles de courbes ont été choisies parce que les programmes correspondants présentent une **invariance d'emboitement** lorsqu'on passe d'une courbe à la suivante dans la même famille : le programme d'un arbre quelconque s'écrit en appelant toujours **de la même façon** le programme de l'arbre précédent .

C'est ce phénomène qui permettra l'écriture récursive. Le programme de l'ARBRE 3 ainsi que le programme récursif de l'arbre de **niveau N** (nombre d'exécutions d'un même appel récursif) dont le tronc a pour mesure C figurent au § 2-3 -1 (Cette séance est décrite dans C . Dupuis, D . Guin , 1988) .

Séance 2 : Essais (Comprendre le fonctionnement d'une procédure récursive)

••

```
POUR ESSAI : N
SI : N = 0 ALORS [ STOP ]
SPA
ESSAI : N - 10
SPB
FIN
```

N est supposé multiple de 10 ,

SPA et **SPB** sont des sous-procédures ou de simples instructions.

••

Analyse de la situation :

C'est sur une idée de P. Mendelsohn (1988) que nous avons fourni aux élèves ce schéma de référence afin qu'ils manipulent des procédures ESSAI : nous avons fait des variations systématiques des sous-procédures SPA et SPB en jouant sur les variables suivantes :

Type de sous procédure :	dessin de dimensions fixes
	dessin dont les dimensions varient en fonction du niveau
	écriture d'un "texte" qui varie en fonction du niveau
Type de récursivité :	récursivité **terminale** (absence de SPB)
	récursivité **non terminale** (absence de SPA)
	récursivité **centrale** (présence de SPA et SPB, qui peuvent être identiques ou différentes).

Nous avons aussi fait quelques variations non systématiques sur le nombre de variables d'entrée dans la procédure (1 ou 2), le type d'appel (ESSAI : N-10 ou N-1), le test d'arrêt . En tout 17 items ordonnés ont été proposés (pour plus de détails sur les questions, on pourra consulter le compte-rendu de notre préexpérimentation C. Dupuis , M.A. Egret , D. Guin , 1985).

Le conflit créé par le passage de la **récursivité terminale** (absence de SPB) à la **récursivité non terminale** ou centrale doit permettre aux élèves de se **construire** une représentation du fonctionnement du dispositif informatique. Les seules tâches demandées étaient une prévision de l'exécution que les élèves devaient noter, puis, si la prévision était fausse, une explication de leur erreur, afin de comprendre le fonctionnement en **confrontant** prévision et exécution. Les différences éventuelles entre prévisions et exécutions étaient toujours suffisamment flagrantes pour être perçues par les élèves comme des contradictions.

Le **besoin de modèles** de fonctionnement de la récursivité a été exprimé par les élèves dès qu'ils ont rencontré la récursivité non terminale. Nous leur avons à ce moment-là proposé différents modèles de fonctionnement afin qu'ils s'approprient celui qui leur permettrait de mieux comprendre le phénomène : un premier modèle mettant en évidence l' **insertion de lignes**, un second modèle qui est un tableau de **simulation de l'exécution** (D.Guin , 1986). Ces 2 modèles de fonctionnement permettent de comprendre la suspension de l'exécution de la procédure appelante. En fournissant ces modèles, nous rejoignons le point de vue d'A. Rouchier (cité par P. Mendelsohn , 1988) qui estime que : " Cet aspect **suspensif** constitue un **obstacle** essentiel à la construction d'une interprétation correcte, il doit donc être explicitement présent lors des séances d'apprentissage ".

Nous avons observé que :

- comme prévu, les 7 groupes d'élèves ont fait une prévision d'exécution fausse pour la première procédure récursive non terminale. Après la présentation des modèles de fonctionnement, 3 groupes ont fait des prévisions correctes, tandis que 4 groupes ont continué à faire un certain nombre de prévisions incorrectes. Au moment où leurs prévisions deviennent correctes, 3 d'entre eux font explicitement référence à l'un des modèles de fonctionnement fourni.
- au plus tard à partir du quatorzième item (sur 17 !), les prévisions de tous les groupes sont correctes.

Séance 3 : Ecriture de procédures récursives

2-3-1 Ecriture récursive des courbes

Nous avons fait une séance de synthèse qui a permis de passer d'une écriture respectant l'invariance d'emboitement à l'écriture récursive des courbes. Voici un exemple de ce passage :

```
POUR  ARBRE3 : C

AV  :  C  TG    45
ARBRE2      :C/2
   TD         90
ARBRE2        :C/2
TG   45   RE   :C
FIN
```

```
POUR ARBRE  : C  : N
SI  :N = 0 ALORS [STOP]
 AV  : C      TG      45
ARBRE      : C / 2 :N-1
           TD         90
ARBRE  :  C  /  2  :N-1
 TG  45       RE      :C
FIN
```

Le passage s'est fait sans difficultés, quoiqu'il s'agisse ici de procédures présentant plusieurs appels récursifs, autrement dit de **récursivité non linéaire**. En effet, l'invariance d'emboitement qui avait été mise en évidence favorise une représentation statique de la procédure, nécessaire à l'écriture récursive .

2-3-2 Ecriture de programmes récursifs

Sept "projets" ont été proposés aux élèves. Ils correspondent aux réalisations de programmes récursifs à récursivité centrale, graphiques ou non graphiques, incluant parfois une procédure avant le STOP. Tous les groupes ont écrit au moins un programme correct , certains très rapidement .

3) Analyse des tests de prévisions

Deux tests individuels ont été proposés aux élèves . Ce sont des tests individuels papier-crayon demandant des prévisions d'exécution et des complétions de procédures récursives.

3-1 - Un critère d'analyse : interprétations de la récursivité centrale

Les interprétations présentées ici ont été rencontrées au moins une fois , mais pas toutes , dans le même exercice. Pour présenter les différentes modalités, nous utiliserons les notations du schéma de référence (voir 2) séance 2) pour indiquer la prévision d'exécution écrite par l'élève pour ESSAI 30 .

Modalités du critère :

R - l'appel récursif central ne décleache pas la suspension de l'exécution de la procédure appelante.

R 0 Les procédures situées après l'appel ne sont jamais exécutées .

R 1 et 2 Les procédures situées après l'appel sont prises en compte lorsque le test d'arrêt est positif et sont alors exécutées :

R 1 : soit pour la seule valeur initiale de la variable

R 2 : soit pour la dernière valeur de la variable avant le STOP.

RFin 1, 2 et 3 Toutes les procédures sont exécutées dans l'ordre où elles sont écrites et pour toutes les valeurs de la variable . Mais la variable change de valeur :

RFin 1 soit après chaque exécution de SPB

RFin 2 soit avant chaque exécution de SPB

RFin 3 soit avant et après chaque exécution de SPB

D A T - La procédure fonctionne comme une succession de Deux programmes récursifs avec Appel Terminal.

C - prévision correcte.

La prévision correcte peut être issue d'un modèle correct, global ou analytique, ou d'un modèle global "miroir" (cf page suivante) .

Prévisions pour ESSAI 30

R0	R1	R2	Rfin1	RFin2	RFin3	DAT	Correct
SPA 30	SPA 30	SPA 30	SPA 30	SPA 30	SPA 30	SPA 30	SPA 30
SPA 20	SPA 20	SPA 20	SPB 30	SPB 20	SPB 20	SPA 20	SPA 20
SPA 10	SPA 10	SPA 10	SPA 20	SPA 20	SPA 10	SPA 10	SPA 10
	SPB 30	SPB 10	SPB 20	SPB 10		SPB 30	SPB 10
			SPA 10	SPA 10		SPB 20	SPB 20
			SPB 10			SPB 10	SPB 30

3-2 - Modèles global ou analytique

R. Samurçay et A. Rouchier (1987) distinguent deux types de modèles :

" - Nous dirons que les élèves utilisent un **modèle global-relationnel** si, dans leur analyse, ils ne font pas appel à la simulation de l' exécution du programme et qu'ils utilisent un système de règles du type: les appels **antérieurs** à l'appel récursif produisent des résultats dans l'ordre **décroissant** de la variable ; les appels **postérieurs** à l'appel récursif produisent des résultats dans l'ordre **croissant** de la variable .

- Nous dirons que les élèves utilisent un **modèle analytique-procédural** si, dans leur analyse, ils font appel à la simulation de l'exécution du programme ". Le contexte dans lequel travaillent R. Samurçay et A. Rouchier est différent du nôtre puisque nous ne demandions pas aux élèves une analyse du fonctionnement mais une prévision d'exécution des procédures.

Les modalités classées sous R et DAT ci-avant correspondent à une simulation et précisent donc les cas où l'on peut affirmer que les élèves utilisent un modèle analytique-procédural . Lorsque la prévision d'exécution est correcte, il se peut que nous ne soyions pas capables, localement , de savoir quel modèle les élèves ont utilisé . Par contre, il existe des cas où il est évident que les élèves ont utilisé un modèle global : lorsque ce modèle global "miroir" conduit à des prévisions incorrectes !

3-3 - Modèle global "miroir"

L'utilisation des seules règles d'action énoncées plus avant peut conduire à une prévision incorrecte. Le schéma de référence ne permet pas de rendre compte de cette conception, car elle ne peut être distinguée d'une conception correcte que dans certaines situations. Il nous faut donc spécifier SPA et SPB . Prenons l'exemple de la procédure suivante :

```
POUR TOTI :MOT
SI VIDE? :MOT  ALORS  [STOP]
ECRIS DER   :MOT
TOTI  SD    :MOT
ECRIS PREM  :MOT
FIN
```

La prévision d'exécution était demandée pour TOTI "SAC. La prévision correcte est **C A S S S S**

Suivant un modèle " **miroir** " , l'élève fournit la réponse **C A S S A C**. On voit que cette réponse satisfait les règles d'actions du modèle global et qu'il ne peut y avoir eu simulation de TOTI. Ce modèle "miroir" n'est pas le résultat des acquis antérieurs à l'enseignement de la récursivité mais bien d'une analyse du fonctionnement de la récursivité.

3-4 - Un critère d'analyse : interprétations du test d'arrêt

Certains élèves exécutent **une dernière fois** la procédure pour la valeur de la variable pour laquelle le test d'arrêt est VRAI. La position de ce test, au début du programme récursif, explique cette erreur. La structure du programme : contrôle puis action, est analogue à celle que l'on rencontre dans les structures itératives du type TANT QUE....FAIRE, dont J.Rogalski et G. Vergnaud (1987) soulignent qu'elles posent plus de problèmes que la structure REPETE.....JUSQU'A qui est plus proche du modèle spontané.

Cette interprétation erronée du test est indépendante de la modalité d'interprétation de la récursivité centrale. On notera qu'elle n'est pas repérable lorsque le test d'arrêt est du type SI : N = 0 ALORS [STOP] et que N désigne la dimension d'un dessin à exécuter. A titre d'exemple, dans la modalité R1 du critère précédent, on observera la prévision ci-contre :

SPA	30
SPA	20
SPA	10
SPA	0
SPB	30

3-5 - Résultats croisés pour deux prévisions du test

Nous ne présenterons ici que les résultats des prévisions pour TOTI "SAC et une procédure ESSAI 30 (voir schéma de référence) où SPA était le dessin d'un "L" dont les 2 branches sont de même mesure N et SPB le dessin d'un carré de côté N .

		ESSAI 30		
		correct	RFin2	DAT
TOTI "SAC	correct	6		
	miroir	3		
	R1	1		
	RFin1 ou 2	6	1	
	DA[1]			1

Dix élèves ont donc fourni des prévisions correctes pour Essai 30 et fausses pour TOTI "SAC. La conception " **miroir** " est évidente pour 3 élèves d'entre eux (car elle suffit pour réussir ESSAI 30) . On peut soupçonner que les 7 autres ont aussi une conception de ce type, mais que les difficultés de TOTI "SAC les font retourner à des conceptions antérieures n'incluant pas l'aspect suspensif.

4) Discussion :

Alors que tous les élèves avaient été capables, dans la séance 3 (2) , d'écrire au moins une procédure récursive correcte pour un "projet" graphique, en interaction avec l'ordinateur, l'étude de leurs prévisions d'exécution montre :

- une grande **variété dans les erreurs**, si l'on considère l'ensemble des exercices que nous leur avions proposé ,

- une grande **instabilité des procédures incorrectes** chez 15 élèves sur 18 .

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[1] **Deux Appels** : nous avons observé que quelques élèves (1 ici) considéraient le programme avec appel récursif central comme la succession des 2 programmes récursifs déjà simulés (SPA ou SPB vides). La situation avait été construite pour rendre visible cette interprétation non pertinente

Nous poursuivons l'étude en analysant les productions des élèves sur l'écriture de procédures récursives. Nous avons d'une part constaté qu'un modèle "miroir" est suffisant pour écrire correctement bon nombre de procédures récursives (possédant une certaine symétrie) . D'autre part , dans l'analyse de la séance 1 (C . Dupuis et D . Guin , 1989) , nous avons observé qu'une représentation du fonctionnement exclusivement en termes d'exécution est un **obstacle** à la mise en évidence de l'invariance d'emboitement . Par contre , une représentation en termes d'états et de relations entre ces états , les relations étant gérées par le dispositif informatique , est efficace . Nous sommes donc amenées à formuler les questions suivantes :

QUESTIONS : quelle compréhension du fonctionnement du dispositif informatique est **nécessaire** pour l'écriture de procédures récursives ? Suffit-il d'avoir un modèle **global** de représentation du fonctionnement ?

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LANGAGE ET REPRESENTATION

DANS L'APPRENTISSAGE D'UNE DEMARCHE DEDUCTIVE

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To analyse how a demonstration works, we must distinguish between surface structure and deep structure. The first one is similar to the surface structure of an argumentation. But its deep structure is quite different : it is based on a proposition substitution that takes into account the status and not the meaning of propositions.

This analysis offers new ways for teaching demonstration processes : specifically proposing deductive organisation tasks on representations about the deep structure, regardless of problem-solving tasks, and asking for the description of this organisation in the ordinary language of students.

We made an experimentation with 13-14 olds students. We present here proof texts of a student whose important evolution is very represensative of the other's progress.

A la suite de Piaget de nombreuses recherches ont tenté d'organiser ou d'évaluer l'apprentissage du raisonnement mathématique à partir d'activités combinatoires et à partir de l'utilisation des connecteurs logiques. L'implication matérielle, qui correspond à la forme mathématique du "si..alors...", s'est trouvé privilégiée dans ces recherches. Devant les impasses et les constats d'échecs enregistrés, une autre approche, centrée sur l'argumentation s'est développée : elle cherche comment les discussions engendrées par de véritables conflits cognitifs peut engendrer des procédures de preuve. Dans cette approche, le langage naturel retouve ses droits pour l'apprentissage d'une démarche déductive.

L'observation du comportement des élèves devant des situations présentant des aspects contradictoires (Duval 1983,1985) et une analyse cognitive du fonctionnement d'une démonstration nous ont conduit à explorer une autre voie. Les difficultés d'une démarche déductive tiennent d'abord aux écarts entre sa structure profonde et sa structure de surface ((Duval & Egret 1989a).

Au niveau de la strucure profonde, le raisonnement déductif fonctionne comme une substitution de propositions, analogue à la substitution des expressions dans un calcul. *Cette substitution s'effectue d'abord en fonction du statut des propositions et non en fonction de leur contenu.* C'est ce point qui constitue l'un des obstacles important à la compréhension de la démarche déductive pour les élèves

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Au niveau de la structure de surface, la démonstration se déroule linéairement comme un texte: les phrases s'ajoutent les une une aux autres et donnent lieu à une appréhension globale. En outre, *les connecteurs logiques qui y apparaissent ne renvoient pas à des opérations de pensée mais fonctionnent comme marqueurs du statut des propositions*. A ce niveau, le raisonnement déductif n'est pas toujours clairement distingué d'une argumentation.

Cette analyse du fonctionnement cognitif d'une démonstration conduit tout naturellement à introduire le raisonnement déductif comme *une activité d' organisation de propositions* qui déplace l'attention du niveau de la structure de surface vers la structure profonde. C'est cette introduction et les résultats obtenus auprès d'élèves de 13-14 ans que nous allons décrire brièvement.

Situation d'apprentisage du raisonnement déductif.

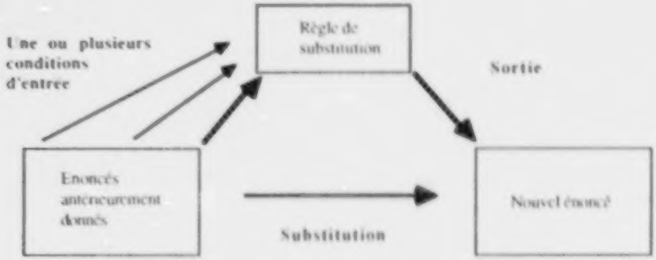
Pour que les élèves soient effectivement placés dans une situation d'apprentisage du raisonnement déductif, deux tâches spécifiques doivent nécessairement leur être proposées.

1) Une tâche de représentation de la structure profonde de la démonstration à faire. Les élèves avaient à construire eux-mêmes un graphe de propositions en utilisant les règles suivantes:

— les hypothèses ne peuvent être que le point de départ d'une flèche vers une autre proposition et jamais un point d'arrivée.

— de la proposition à démontrer aucune flèche ne peut partir

— le passage d'une hypothèse à une conclusion se fait par une proposition-règle, comme un théorème ou une définition.


Une ou plusieurs conditions d'entrée
Règle de substitution
Sortie
Enoncés antérieurement donnés
Substitution
Nouvel énoncé

Le recours à un graphe pour représenter la structure profonde est une procédure qui a été utilisée dans de nombreuses recherches , non seulement pour la démonstration, mais aussi pour la compréhension de textes ou d'histoires (Anderson 1982,1987, Gaud&Guichard 1984, Quillian1969, Rumelhart 1975, Duval 1987..). Mais, généralement, cette procédure est utilisée à des fins heuristiques et non pas exclusivement comme une tache d'organistion déductive des propositions. Le graphe est souvent introduit comme une procédure de recherche (Anderson 1987, Gaud&Guichard1984). Nous, nous l'avons introduit , au contraire, *après une phase de recherche* et après une mise en commun sur la phase de recherche. Les élèves ayant déjà une idée de la démonstration, l'élaboration du graphe devient une tâche spécifique d'organisation déductive des propositions: il devient pour les élèves un outil de contrôle facile et sûr pour distinguer le statut donné à chaque proposition.

2) Rédiger un texte qui traduise l'organisation déductive représentée.

Cette deuxième tâche est aussi importante que la première. D'une part l'expression personnelle dans le langage naturel reste le lieu privilégié où s'accomplit toute prise cde conscience des opérations spontanément faites, ainsi que Piaget l'avait souligné dans ses premiers travaux (Piaget 1967). Il s'agit, ici, que chaque élève établisse les correspondances entre l'ordre de substitution représenté et les expressions qui, pour lui, seront signifiantes du fonctionnement de la démonstration. D'autre part, la pleine maîtrise d'une activité ou d'une démarche ne peut être atteinte que lorsque que le sujet peut en objectiver les résultats dans deux registres différents : une bonne conceptualisation suppose, en effet, que l'on surmonte les difficultés liées aux phénomènes de non-congruence sémantique entre différents registres de représentation ou d'expression (Duval 1988a).

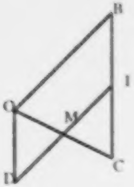
Pour cette deuxième tâche aucune consigne particulière de rédaction n'est donnée aux élèves, autre que celle d'expliquer la démonstration représentée. Et aucun texte corrigé de demonstration n'a été présenté aux élèves au terme de leur travail sur un exercice. On remarquera ici, qu'il ne s'agit pas de lire et d'interpréter une représentation graphique proposée par l'enseignant ou par un autre élève, mais d'expliquer la représentation que l'on a produite. La lecture d'une représentation graphique toute faite serait une tâche différente.

L'expérience s'est déroulée pendant une vingtaine de séances , durant un trimestre, dans une classe de 27 élèves (Egret&Duval 1989b).

Résultats

La comparaison entre les démonstrations produites au début de l'expérience et celles produites quelques semaines plus tard, pour un même élève, montre l'importance de l'évolution accomplie. En voici un exemple. Une évolution analogue a été constatée chez les deux tiers des élèves de la classe.

Exercice 1:
O,B,C sont trois points non alignés
I est le milieu de [BC] et D le point
tel que ODIB soit un parallélogramme.
Pourquoi M, milieu de [ID] est-il le
milieu de [OC]?



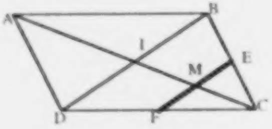
Texte produit:

DO = IB	*DO // IB*
IB = CI	*CI // IB*
DO = CI	*DO // CI*
CD // IO	*CD = IO*

Les diagonales d'un parallélogramme se coupent en leur milieu I. DOIC parallélogramme donc *elles se coupent en leur milieu.*

Quelques semaines plus tard , le même élève produit, pour un exercice similaire, le texte suivant

Exercice 5:
ABCD est un parallélogramme.
I est le point d'intersection des
diagonales, E est le milieu de
[CB]et F celui de [CD].
Les droites (AC) et (EF) se coupent
en M. Montrer que M est le milieu
de [EF].



Texte produit : *Pour trouver qu'un point est le milieu de deux segments cela peut être les diagonales d'un parallélogramme. IL SUFFIT QUE JE PROUVE que IE // FC et IF // CE.*

Il suffit d'appliquer *le théorème des milieux dans le triangle DBC.* On sait que *E est le milieu de BC MAIS IL NOUS FAUT UN AUTRE MILIEU . Ce sera I milieu de DB* puisque *I est l'intersection des diagonales d'un parallélogramme et qu'elles se coupent en leur milieu.* Donc *on peut appliquer le théorème des milieux. Dans le triangle DBC, la droite qui passe par le milieu d'un côté et qui passe par le milieu du côté opposé, cette droite est parallèle au troisième côté. JE SUIS SURE QUE IE // FC.*

MAINTENANT JE FAIS le théorème des milieux pour que IF // EC. On sait que *I est le milieu de DB (voir plus haut) dans le triangle DBC.* On sait que *F est le milieu de CD PUISQUE ILS NOUS LE DISENT.* Alors *la droite qui passe par le milieu d'un côté et qui va au milieu du côté opposé, cette droite est parallèle au 3° côté.* Donc maintenant je sais que *IF // EC et IE // FC donc c'est un parallélogramme.* Et puisque *les diagonales d'un parallélogramme se coupent en leur milieu* alors *M est le milieu de [EF]*

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On remarque tout de suite que le premier texte est à la fois peu structuré et que des propositions essentielles pour la démonstration sont passées sous silence ou n'appraissent pas à leur place:

— la proposition à démontrer est omise,

— la différence de statut entre les différents propositions mentionnées est omise. Par exemple "DOIC est un parallélogrammmme" n'apparaît pas comme conclusion des hypothèses. DO//CI est présenté comme s'il s'agissait d'une hypothèse.

Ce texte juxtapose en fait des propositions. L'unique occurrence de "Donc" intervient avant la dernière proposition. Nous avons souvent relevé que les élèves employaient ce terme "donc" pour tranformer une suite de remarques ou d'observations sur une figure en un raisonnement déductif.

Le deuxième texte apparait tellement différent du premier qu'il peut sembler difficile de les attribuer au même élève sur une période aussi courte. Rappelons qu'aucun texte corrigé n'a été présenté aux élèves, entre-temps, comme modèle. Trois aspects frappent particulièrement dans la rédaction de l'élève:

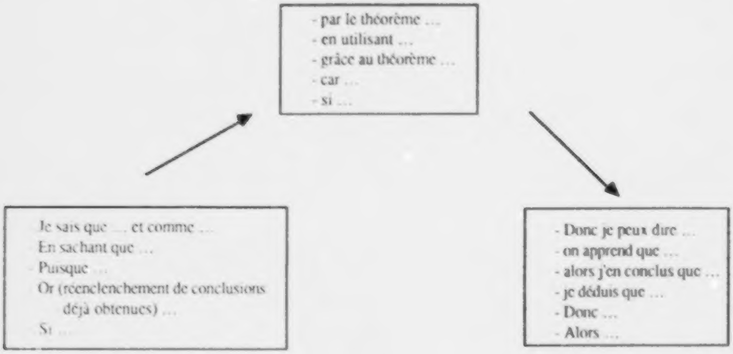
1— Chaque proposition est introduite par une expression qui souligne son statut dans le déroulement de la démonstration. Par exemple, pour les hypothèses: "on sait que....puisqu'ils nous le disent ". Pour les règles de substitution : "puisque....". Pour les conclusions: " Je suis sûre que..", "donc maintenant je sais que...". Ces dernières expressions sont particulièrement remarquables : elles soulignent la prise de conscience d'un gain de nécessité obtenu pour les propositions déduites par substitution. Cela illustre d'ailleurs un phénomène plus général que nous avons retrouvé dans d'autres textes produits par d'autres élèves : *la prise de conscience de ce qu'est le raisonnement déductif s'est spontanément traduite par l'emploi varié d'expressions d'attitudes propositionnelles* et non par le recours à des connecteurs logiques. Cette prise de conscience correspond à un déplacement de l'attention du contenu des propositions vers le statut et à la découverte que l'organisation déductive de chaque pas de démonstration repose sur le statut des propositions. Bien qu'ayant un caractère plus subjectif que les connecteurs logiques, les attitudes propositionnelles permettent d'exprimer la différence modale de certitude attachée aux différents statuts qu'une propostion peut avoir selon les contextes théoriques où elle apparaît.

2— Il y a prise en charge des énoncés par le locuteur. On a quelquefois souligné le caractère impersonnel du discours des élèves lorsqu'il s'agit de produire des textes scientifiques. Nous voyons ici l'élève ne pas hésiter à s'exprimer à la première personne. Et ce qui est encore plus significatif est l'opposition entre l'emploi de l'impersonnel "on" et celui du personnel "Je" : *"on" est réservé pour introduire les hypothèses qui lui sont données et "Je" apparait pour introduire les conclusions que , lui , l'élève, dégage*. Cela confirme qu'il y a bien eu prise de conscience de ce qu'est le raisonnement déductif.

3— Un discours explicatif soulignant les démarches qui vont être effectuées apparait. Par exemple effectuer un pas de démonstration pour obtenir une condition nécéssaire à l'application d'un théorème: " mais il nous faut un autre milieu...". C'est une appréhension globale de toute la démonstration qui s'exprime avec ce discours explicatif.

Entre ces deux textes produits par le même élève, à quelques semaines d'intervalle, il s'est produit une évolution très importante. Cette évolution est marquée par le franchissement de plusieurs seuils que nous avons pu observer chez la plupart des élèves. (Egret &Duval 1989b). *Or le franchissement de ces seuils s'est faite* non pas dans la tâche de rédaction mais *dans celle de contruction de la représentation de la structure profonde* qui était demandée pour chaque problème. Cela indique que pour les démonstrations, en géométrie, la construction du graphe des propositions apparait comme l'intermédiaire nécessaire (au moins transitoirement) entre la figure et le texte (Duval 1988b).

L'évolution que nous venons de décrire pour un élève est représentative de l'évolution accomplie par les deux tiers des élèves de la classe. Après une phase de découverte, nous avons vu apparaître aussi le recours à des connecteurs logiques ou argumentatifs. Le schéma ci-dessous donne un petit échantillon de la diversité des expressions rencontrées dans les textes d'autres élèves pour indiquer le statut des différents propositions.


- par le théorème ...
- en utilisant ...
- grâce au théorème ...
- car ...
- si ...
- Je sais que ... et comme ...
- En sachant que ...
- Puisque ...
- Or (réenclenchement de conclusions déjà obtenues) ...
- Si ...
- Donc je peux dire ...
- on apprend que ...
- alors j'en conclus que ...
- je déduis que ...
- Donc ...
- Alors ...

Conclusion

L'entrée dans le raisonnement déductif ne se fait pas par la manipulation d'opérations du type de l'implication matérielle. Celle-ci n'est d'ailleurs pas la seule forme logique d'implication. L'implication matérielle joue aussi un rôle important dans le raisonnement mathématique et ne présente pas autant de difficultés pour les élèves. Il faudrait prendre aussi en compte l'implication stricte, de caractère modal, et qui est plus proche du raisonnement propre à la pensée naturelle.

Le raisonnement déductif est d'une nature différente de l'argumentation spontanément mise en oeuvre dans des discusions ou dans des débats relatifs à des conflits cognitifs. Une argumentation ne fonctionne pas d'abord sur le statut des propositions mais sur leur contenu. La prise en compte du statut des propositions n'y est pas essentielle. Cela permet d'ailleurs à l'argumentation d'être une phase privilégiée de raisonnement dans les phases de recherche sur un problème. En outre les argumentations donnent lieu à des représentations différentes de celles de la démonstration: ce ne sont pas des graphes orientés commme des arbres, mais des réseaux à relations multiples et non orientées entre les noeuds. La représentation de la structure profonde d'une argumentation fait apparaître qu'il n'y a pas de pas d'argumentation.

Le développement de "la pensée formelle" ne peut donc pas être décrit et expliqué à partir des activités combinatoires et des opérations de la logique propositionnelle. Une telle conception conduit en effet à méconnaître l'importance de l'interaction entre les représentations non-discursives qu'un sujet peut lui-même construire et les pratiques discursives qui lui sont spontanées. Une telle interaction ressemble plus à une activité de codage ou de recodage qu'à une discussion "intériorisée". C'est d'ailleurs pourquoi elle offre à chaque sujet un moyen de contrôle indépendant du jeu de la contradiction ou du consensus dans la communication : ce contrôle est inséparable d'un processus d'appropriation personnele du sens.

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TOWARDS A THEORY OF TRANSITION

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Analysis of data from a longitudinal study of the effects of the transition from elementary to secondary schooling on attitudes towards, and achievement in, mathematics is summarised. Through the two case studies described, we draw attention to the opportunity which mathematics teachers have in capturing and maintaining the interests of students beginning their secondary schooling, and point out the fragile balance between attitude and ability. Difficulties in presenting an adequate theory of transition are discussed.

A Transition Problem or a Transition Opportunity?

The importance of the effects of the transition from elementary to secondary school on attitudes towards, and achievement in mathematics has long been recognised (Power & Cotterell, 1981). Traditionally the issue has been presented as a "transition problem." In the Cockcroft Report on mathematics education in Great Britain, for example, the committee stated: "We believe that the greatest problems exist in the transfer to secondary or upper school" (Cockcroft, 1982, para. 429).

There have been a number of studies of the so-called transition problem (e.g. Trebilco, Atkinson & Atkinson 1977; Devlin & Goodger 1978; Power & Cotterell 1981; Clarke 1984a). However, in some recent longitudinal investigations, data has been presented which seems to challenge the notion of a transition problem (Clarke, 1985; Ellerton & Clements, 1988). We have found that many (indeed a majority of) children start secondary mathematics with positive feelings about the subject and a desire to do well. Students we have interviewed have told us that they are now "loving" mathematics, that it is "much better than last year," and they feel that they now understand it better (Ellerton & Clements, 1988, p. 134). Yet, by the end of their first year of secondary school, many of these same students have come to believe not only that they are doing poorly at mathematics, but also that they probably never will do well at it. Thus, the "transition opportunity" has given way to a "transition problem." It seems to us that this observation presents a major challenge to mathematics educators: first, there is a need for research to investigate whether it is generally true around the world; and second, there is a need, if it is true, to reverse the trend so that the opportunity is not lost.

Two major recent studies which take account of the elementary school/secondary school transition effect have been reported by Clarke (1985), and by Ellerton and Clements (1988). While these studies were designed and carried out independently, both used ethnographic approaches, and both developed and analysed longitudinal data sets. Clarke studied the changes in students' mathematical attitudes and achievements over a three-year period (as students proceeded from Grade 6 to Grade 8), and Ellerton and Clements studied changes over a seven-year period (as students moved from school entry (Preparatory Grade) through Grades 1 to 7. Both studies took place in Victoria (Australia) where the elementary school/secondary school transition occurs between Grades 6 and 7; and both investigations focused on the differential reactions of individual students from *different* elementary schools, who came together in the *same* Grade 7 secondary school mathematics classroom. These students had come from elementary schools where the approaches to teaching mathematics varied considerably. But at secondary school, they had the same mathematics teacher, and they studied mathematics together in the same classroom.

The Clarke study. Clarke (1985) monitored the individual progress, from Grade 6 to Grade 8, of ten students (who originally came from four different elementary schools) through observations, questionnaires, interviews and tests. He also attempted to build up a composite picture of the secondary mathematics classroom environment experienced by the ten children. He developed a procedure which enabled teachers to monitor and respond to the changing needs of students: Grade 7 students were regularly (once every two to four weeks) given the opportunity to inform the teacher of difficulties experienced, success achieved, and sources of anxiety, and were encouraged to reflect critically on the teaching and learning of secondary mathematics (p. 256).

Clarke maintained his data suggested that "the impact of secondary mathematics on a student's mathematical behaviour may be determined during the first year of secondary school through the evolution of community opinions which the child comes to share" (p. 255). He went on to suggest that there is a need for a more comprehensive descriptive framework than the usual restricted consideration of attitudes and achievement if the *process* of secondary schooling and secondary mathematics is to be usefully understood. From his case study data, Clarke argued that variety in student background, differences in personality, and evident diversity in students' responses to the same mathematics classroom do not suggest that a single teaching style can meet the needs of children commencing secondary mathematics (p. 255).

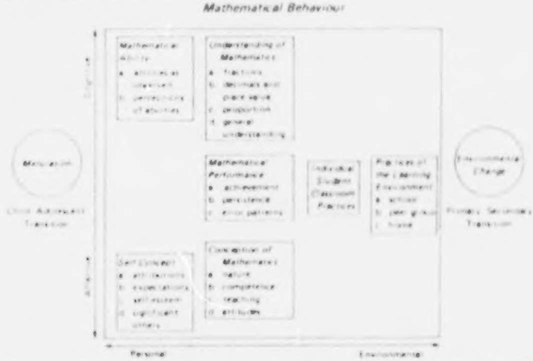
Mathematical Behaviour
Mathematical Ability
b. perceptions of abilities
Understanding of Mathematics
a. fractions
b. decimals and place value
c. proportion
d. general understanding
Motivation
Child-Adolescent Transition
Mathematical Performance
a. achievement
b. persistence
c. error patterns
Individual Student Classroom Practices
Practices of the Learning Environment
a. school
b. peer group
c. home
Environmental Change
Primary-Secondary Transition
Self Concept
a. attributions
b. expectations
c. self esteem
d. significant others
Conception of Mathematics
a. nature
b. competence
c. teaching
d. attitudes
Personal
Environmental

Figure 1. The elements of a model of mathematical behaviour (from Clarke, 1985, p. 232)

Clarke (1985) presented a two-dimensional model (see Figure 1) which, he claimed, was entirely a product of the research he had carried out (p. 232). This model attempted to describe and explain the mathematical behaviour of Grade 7 students: one dimension had poles labeled "personal" and "environmental", and the other, "affective" and "cognitive". Various factors influencing mathematical behaviour (e.g. mathematical ability, conceptions of mathematics, and practices of the learning environment) were placed at appropriate positions.

Clarke's model provides a useful starting point for discussion of the transition opportunity/problem. There are at least three issues which seem to arise from it: (a) Are the two dimensions of the model independent? (b) What traits are represented by the two dimensions? (c) What single continuum might have "affective" and "cognitive" poles?

The Ellerton and Clements study. Since much of the discussion later in this paper will be based on data obtained in this on-going study, it will be described in greater detail than was Clarke's. In 1981, over 500 children who commenced their schooling at nine schools (seven state,

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one Catholic and one independent), in different socio-economic regions in and around the city of Geelong, were given several language tests, and a wide range of background data (especially in relation to their home environments) was also obtained. Further data, concerned with their acquisition of reading and mathematical skills, were obtained in 1982 and 1983, and in 1985, 1986 and 1987 various aspects of their mathematical and language development were investigated by means of interviews and pencil-and-paper tests. Also, in each of the years 1985, 1986 and 1987, the children were invited to make up mathematical problems of their own (Ellerton, 1986).

At the end of 1987, over 300 of the original 500 children were still attending the nine primary schools in the Geelong region, and in 1988, almost all these children transferred to Geelong secondary schools. We followed 90 of the children (our 'transition sample') to nine secondary schools (seven state high or technical schools, one Catholic school, and one independent school). Early in 1988 these students were given a range of pencil-and-paper mathematical problems, several attitudinal instruments (including a Likert scale, a projective task, and an instrument designed to measure students' confidence in their mathematics procedures); students were also invited to describe, in writing, the school mathematics which they were doing in 1988, and also to outline their method for solving two given mathematical problems. One-to-one interviews with all 90 students were audio-taped, as were interviews with their parents, and with their mathematics teachers. Further pencil-and-paper and interview data were obtained towards the end of 1988.

The analysis presented in this paper incorporates considerably more data than that used in our initial report of the study (Ellerton & Clements, 1988). This initial report presented a summary of the data for just two children, Cathy and Peter (not their real names), who attended different elementary schools but who were in the same class at the same secondary school in 1988. Cathy and Peter were selected because although their scores on standardised mathenatics tests had been virtually identical in Grades 4, 5 and 6, the mathematics programs which they had experienced in their elementary schools were significantly different.

Summary of data for Cathy and Peter (to June, 1988). Table 1 analyses and interprets data for Cathy and Peter to the middle of their first year in secondary school.

At the beginning of 1988, Grade 7 mathematics for both Peter and Cathy promised to provide an opportunity for a new start in the subject. Peter had always wanted to do well in mathematics (as had his older brother and sister), but in elementary school he had experienced increasing difficulty with the subject. This had been compounded by a month's absence from school, through illness, in Grade 6. His initial enthusiasm for Grade 7 was somewhat dampened by the fact that he was required to study, yet again, the topic of fractions which he had never really understood in elementary school. In an interview early in Grade 7 he commented that it had taken him " a period or so to get to know fractions well." Such a statement is indicative of his over-confidence in his ability to do mathematics. Our data indicate that even at the end of Grade 7 he had not mastered elementary fraction ideas. For example, at that time, he indicated that he was *certain* that "15/16 = 45/46" and he *thought* that "1/3 was less than 1/4."

Although early in her Grade 7 year Cathy thought mathematics was important and wanted to do well in it, she lacked confidence in her own ability to get correct answers, even when she knew how to solve problems. She blamed her Grade 4 teacher for not covering appropriate mathematics, and thought that this had forced her to have to cram too much new work into her head in Grades 5 and 6. She thought she would succeed in secondary school mathematics, as had her older sister.

Table 1.
Comparison of Data on Mathematics Performance and Attitudes of Cathy and Peter, Covering Grades Prep - 7

Aspect of Performance/Attitude	*Cathy*	*Peter*
1. Language facility, Grades Prep-6 (Standardised comprehension tests)	Above average (similar scores to Peter)	Above average (similar scores to Cathy)
2. Performance on standardised mathematics tests (Grades 4-6)	Strong (almost same as Peter) and above average for her class.	Strong (almost same as Cathy) but below average for his class.
3. Level of confidence in answers to mathematics questions (at beginning of Grade 7).	Low: even when she knew how to solve problems, she was not confident of her answers.	High: even when he appeared to have little idea of how to solve problems, he was confident his answers were correct.
4. Attitudes towards mathematics:		
(a) Grades 4 and 5.	Felt she was not coping.	Felt he was good at maths.
(b) Grade 6.	Doing better, but was not confident; had lost interest by end of year.	Desperately wanted to do well.
(c) Early in Grade 7.	Keen to succeed, but bored (says she has done it all before).	Hated fractions, frustrated.
(d) Mid Grade 7	More confident.	Hated fractions, highly competitive.
5. Expectations of secondary school mathematics (at end of Grade 6).	Looked forward to a new beginning ("It won't be hard - it will get harder during the year. But it won't be because they teach you how to do it.")	Hoping to do well. ("Sort of looking forward to it. I want to know what they're doing first ... but if it's too different ... that's what I'd be worried about.")
6. Prepared to seek help in mathematics (Grade 7).	Yes, both in class and at home.	Reticent in class; older sister helps at home (if available).
7. Parents' perception of the importance and relevance of mathematics (early in Grade 7).	"Maths should be practical. It's not practical in Grade 7. It gets more practical in Grades 8 to 10 because they start to use calculators and short cuts."	"You have to do very well at maths or English."
8. Problem-solving skills:		
(a) types of mathematical problems created on request.	Creative; problems made up often had several operations or were carefully chosen to make them difficult.	Made up problems with single operations; large numbers used to make problems difficult. Found own problems difficult to describe.
(b) solving set mathematical problems.	Reasoned clearly (written and verbal) when solving problems.	Easily confused and found it difficult to tackle problems in a systematic way.

Table 1. (cont'd.)

Comparison of Data on Mathematics Performance and Attitudes of Cathy and Peter, Covering Grades Prep - 7

Aspect of Performance/Attitude	*Cathy*	*Peter*
9. Teacher's perception of student's attitude and progress in mathematics: (a) Early Grade 7	Capable, but tended to grizzle - wanted teacher to know that she could really do most things.	"Not terribly brilliant at his work, but making an effort."
(b) Mid Grade 7	Doing quite well, but careless. Coping without difficulty; asked for help when needed.	Very quiet; did not ask many questions. Wanted to do well. Easily confused.
10. Child's perception of: (a) The nature of secondary school mathematics (Grade 7).	Prefered new work to revising work she already knew. Liked a challenge. Did better in Grade 7 "because last year we did harder things." Found fractions easy but didn't enjoy them as much in Grade 7 ("because I already knew them").	Liked doing different things and spending longer on them. ("If you keep doing the same thing for a period or two, you know how to do it pretty well . . . You've got to do maths, so you might as well do it and get on with it.")
(b) The importance of mathematics (Grade 7).	Enjoyed mathematics and thought it was very important.	"It could be useful later in life when I get a job. But I'm not in a shop or anything *now*."
(c) The need to do well at mathematics (Grade 7).	Not as competitive as Peter. Looked for confirmation of correct work.	Fiercely competitive.
(d) What happens in the mathematics classroom (Grade 7).	"He (the teacher) usually explains, and writes different things up on the board, then just gives us work out of the book, or he does it up on the board and we copy it down."	"You get a bit sick of copying off the board. You'd be better to work from the book a bit more."
11. Parents' perception of child's approach to mathematics (Grade 7).	"She is above average, but she is not at the top. If she takes her time she is fine. She always *says* she doesn't understand, but she *does*."	"He likes to be at the top in everything. He is not content with only a pass . . . He is average or a little better, at maths."
12. Parents' perception of the role of teachers (Grade 7).	"If the teacher loves maths, it will come across as more interesting to do than a teacher who hates maths and is more-or-less teaching it because they have to."	"If you like the teachers, then you are half way there. Teachers need to be available whenever a child needs help - for example, in lunch-hours."

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In Ellerton and Clements (1988, pp. 139-140) we commented that about three months before the end of his Grade 7 year, Peter "was easily confused in mathematics classes," that "the novelty of secondary school was wearing off," and that the opportunity for Peter to become good at mathematics was "fading away." For Cathy, on the other hand, Grade 7 mathematics had been a time of opportunity when she had been able to prove to herself that she could do mathematics. With elementary school mathematics behind her, she was no longer inclined to blame any difficulties she now had in mathematics on earlier events. She had experienced a new beginning.

The analysis of the data for Cathy and Peter and the large data set on the other students in the study, mocked the common practice of placing beginning secondary (Grade 7) students into mathematics groups on the basis of scores on standard achievement tests. We concluded (Ellerton & Clements, 1988, p. 140) that such a procedure is a recipe for lost opportunity. The data drew attention to the fragile balance between attitude and ability, and the interaction of these with the mathematics curriculum and classroom experiences. We observed that, as the initial excitement generated by the novelty of secondary school diminished, the opportunity for Peter to become good at mathematics seemed to fade. But, at the same time as Peter was beginning to struggle with Grade 7 mathematics, Cathy took up the challenge, began to do well in the subject, and gained more confidence in her ability to cope. At the conclusion of our 1988 paper we ventured to predict that Cathy would do well in future mathematical studies, and that Peter would not. Yet at the start of Grade 7, both had been performing at approximately the same level in mathematics, and both had been looking forward to secondary school mathematics.

We now extend our analysis by considering end-of-Grade 7 data for Cathy and Peter. We shall then comment on the difficulties inherent in the idea of proposing any theoretical model for explaining mathematical behaviour.

What Might Constitute an Adequate Model of Mathematical Behaviour?

Cathy and Peter: End-of-Grade 7 Data

Cathy. When Cathy was interviewed at the end of Grade 7 she said of mathematics that it was "harder, but better". She added: "Actually, it's not really that hard once you understand it." Cathy's parents said that she had settled into secondary school very well and, as far as mathematics was concerned, was now much more relaxed. Her Grade 7 mathematics teacher commented that she had coped very well during the year, and "shouldn't have any problems in the future." He went on to say that "she's a nice girl, but grizzles about anything."

Performance and attitudinal data collected at the end of Grade 7 indicate that Cathy then had a much better grasp of basic mathematical concepts, was able to solve mathematical problems more readily, and was very much more realistic in recognising whether she could or could not do a problem than at the beginning of Grade 7, when she had lacked confidence and had often thought she might be wrong, even when she had actually obtained correct answers. At the end of Grade 7, however, she knew when she had obtained a correct answer, and she knew when she had not. Grade 7, for Cathy, had been a time when she had become more competent, more confident, and more relaxed so far as mathematics was concerned. Whereas a year earlier it was not clear how Cathy would respond to secondary school mathematics, she now had an excellent foundation for future success in the subject.

Peter. The earlier data indicated that, at the beginning of Grade 7, Peter was reasonably competent at mathematics, very competitive, and desperately wanted to do well at secondary mathematics. Howerer he was over-confident in his ability to obtain correct answers to mathematical problems, and had a strong dislike for fractions. Unfortunately, the study of fractions was an important component of his early secondary mathematical experiences and this, no doubt, contributed to a rapid decline in Peter's mathematical performance and attitude. This prompted us to suggest (Ellerton & Clements, 1988, p. 140) that the trend might not be reversible.

With about four months remaining in his Grade 7 year, Peter and his parents were sufficiently concerned about his lack of progress in mathematics that they decided to hire a private mathematics tutor.This tutor saw Peter one evening each week and, during this time, went over the mathematics which was being studied in class. Peter commented that the tutor helped him just before tests, and added that "because my parents don't know a lot about it, and my sister's away studying, I need the tutor because no-one else at home can help me." "If I get really, really high up in the class," he went on to say, "I probably won't need a tutor any more." Despite these comments, Peter obtained a very low score (5 out of 17) on a test of mathematical understanding administered to his Grade 7 class towards the end of the school year; Cathy, in the same class, obtained third-highest score (14 out of 17). Notwithstanding his low score on this test, Paul was *certain* that he was correct for 14 of his 17 answers, and *thought* he was correct for two of his other answers. By contrast, Cathy was *certain* she was right for only five of her answers (and in each of these cases she was, in fact, correct), and *thought* she was correct for eight of her other answers (she also gave correct answers for these eight). She had been accurate in not thinking she was correct for her three incorrect responses. Peter's over-confidence had not diminished. It seems that despite help from his tutor, he did not know that he did not know. Cathy, on the other hand, knew when she knew.

At the end of the Grade 7 year, Peter's mathematics teacher said that "Peter tends to think that he's got things perfectly under control, and then blows it, totally . . . Even though he shows evidence of having done some extra work and makes an effort, he can be totally confused." This assessment is entirely consistent with our data. So, too, was the teacher's end-of-year assessment of Cathy: "She's coping well, and shouldn't have any problems."

Difficulties Associated with Postulating a Model of Mathematical Behaviour

While Clarke's (1985) model (see Figure 1) raises many interesting issues, we find some difficulty in relating Cathy's and Peter's cases to it. Any complete model of mathematical behaviour would need to explain not only how (a) *cognitive* strategies, (b) *affective* characteristics, and (c) *environmental* factors are inter-related, but also (d) how each of these is related to *mathematical performance*. The above discussion, and that in our previous paper (Ellerton & Clements, 1988), suggest that each of the four categories (cognition, affect, environment, and performance) are multi-dimensional.

We would certainly not expect that any two of the categories were orthogonal. Furthermore, other categories suggest themselves for inclusion in any useful model of mathematical behaviour - like, for example, mathematical ability (however this may be defined), mathematical maturity (Piaget comes to mind), and type of mathematics under consideration (e.g. geometry, analysis). Even if attention were confined to the four categories listed in the last paragraph, it would not be easy to achieve an ordering within any one category - for example, within *environmental* factors, how should one order cultural relevance of the mathematics being considered, the influence of parents, the availability of learning aids in the classroom or at home, and teaching methods?

It is not our intention to be unduly pessimistic. We believe that the transition research by Clarke (1985), and by ourselves, has drawn attention to the opportunity which mathematics teachers have in capturing and maintaining the interests of students beginning their secondary schooling. It seems to us that the notion of transition *opportunity* is much more powerful than the more commonly heard concept of transition *problem*. Clarke's (1984b, 1987) innovative attempts to monitor important aspects of children's mathematical learning provide promising and practical procedures for helping classroom teachers make the most of the opportunities with which they are presented. But we are a long way from being able to present useful models which will enable the mathematical behaviour of teenagers to be interpreted systematically.

A comment made, at the end of the school year, by Cathy's and Peter's Grade 7 mathematics teacher encapsulates the arguments presented in this paper. He said of his Grade 7 class:

> They're a bright, chirpy, happy group, noisy as blazes. The room's terrible - it's a portable. There are a few quite competent kids; most are average to poor. A few are quite troublesome, behaviour-wise. They can really create problems . . . Some kids still run around in class like a blowfly on a bottle. This particular class is the worst Grade 7 I've had for a while. It's much easier to teach History and Geography, where they just have to copy stuff from the board, than it is to teach Maths, where the range of abilities creates big problems. What you put on the board in Maths lessons is not work, but it's a preparation for *doing* work. There's a lot of stress associated with teaching Maths.

A careful and thoughtful reading of this statement, made by an experienced, successful and caring teacher, suggests many subtle variables and associated interactions which need to be covered in any realistic model explaining students' mathematical behaviour. Theory-driven research in mathematics education will be useful only insofar as theories proposed are able provide a reasonable approximation to reality. Our research, and that of Clarke (1985), aims at identifying important variables which will be useful in the quest to generate an adequate theory of transition.

Acknowledgements

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VALIDATION D'UN LOGICIEL D'AIDE A LA RESOLUTION DE PROBLEMES ADDITIFS.
(Evaluation of an aided Additive Problem Solving Software)

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SUMMARY

SIPOS is a software that has been designed to help children improve additive and subtractive problem solving. SIPOS helps (a) identifying Combine, Change and Compare relationship and (b) writing correct operations that match identified problem structures.

Teaching with SIPOS was tested by training two groups of third grade children. Results show that the experimental group still gave good performance with a test that was delivered one month after the learning set: experimental group obviously acquire the provided knowledge.

Otherwise, although small positive differences occur between experimental and control group, pretest and pos-test comparisons indicate difficulties to transfer what has been learned from extra-school training to school training.

1. INTRODUCTION

La validation d'un nouvel outil d'enseignement est toujours une entreprise difficile tant pour la mise en oeuvre que pour le choix des critères d'évaluation des acquisitions. Nous rapportons et discutons ici la mise en place de la validation du logiciel SIPOS, logiciel d'aide à la résolution de problèmes additifs.

Nous avons travaillé sur deux classes de CE2, soit 39 élèves. Chacune a été partagée en deux, une demi-classe, le groupe expérimental, était soumis à l'entrainement par le logiciel, décrit plus bas, l'autre, le groupe contrôle, était simplement soumis à la résolution des mêmes problèmes dans un environnement équivalent (micro-ordinateur). Toutes les séances d'entrainement furent individuelles, s'étalant sur un mois environ. Tous les élèves ont passé en situation de classe un pré-test et un post-test consistant en la résolution de problèmes similaires à ceux présentés dans le logiciel.

2. LE LOGICIEL

2.1. Objectif

L'objectif du logiciel SIPOS est d'entrainer l'élève du cycle élémentaire (2° et 3° grade) à la résolution de problèmes en lui apprenant à identifier la relation additive sous-jacente à l'énoncé, puis à écrire l'opération numérique qui traduit cette relation.

2.2. Cadre théorique

2.2.1. Classification des problèmes

Les travaux de Vergnaud (Vergnaud 1982; Vergnaud et Durand, 1976) montrent que le principal facteur de difficulté des problèmes additifs réside dans le calcul relationnel, et non dans le calcul numérique. Ceci conduit à une classification des énoncés en fonction de la nature de la relation. On retiendra les trois relations primitives suivantes :

- Parties/Tout (ou composition de mesure) : Pierre a 5 francs. Marie a 3 francs. Combien d'argent Marie a-t-elle?
- Transformation (ou changement d'état) : Pierre a 5 francs. Marie donne 3 francs à Pierre. Combien d'argent Pierre a-t-il maintenant?
- Comparaison : Pierre a 5 francs. Marie a 3 francs de plus que Pierre. Combien d'argent Marie a-t-elle?

Les trois problèmes cités se résolvent par la même addition, mais les significations relationnelles sous-jacentes à l'opération arithmétique ne sont pas équivalentes. Les auteurs américains (Capenter, Hiebert et Moser, 1981; Riley, Greeno et Heller, 1983) parviennent à la même classification sous les termes de Combine, Change et Compare. Pour chacune de ces trois classes de problèmes, le principal facteur de difficulté ne concerne pas le sens de la relation (positive ou négative), mais la place de l'inconnue dans cette relation.

2.2.2. Enseigner les relations

Les relations Parties/Tout, de Transformation et de Comparaison qui permettent de catégoriser les structures additives ne sont pas enseignées. Or l'enfant ne peut, semble-t-il, construire seul ces relations. Selon la théorie piagétienne, la conceptualisation s'élabore à cet âge sur les propriétés des objets et non sur les rapports entre les objets.

En l'absence de cet enseignement des relations, l'élève n'aborde pas le problème à résoudre avec l'idée d'identifier le problème à une classe relationnelle, mais avec l'idée de trouver "l'opération à faire". Succintement, il tente de passer directement de l'énoncé à l'opération, et pour cela, s'appuie sur certains mots-clef de l'énoncé (*il reste, gagner, perdre, ensemble, de moins que,* etc.), jouant le rôle de mots inducteurs d'opérations. Notre objectif est de briser ce type de fonctionnement direct en imposant l'identification de relations. Il s'agit d'aider l'élève à reconnaître, à travers l'énoncé d'un problème, la structure relationnelle de ce dernier, ce qui suppose de dépasser les expressions linguistiques, sémantiques pour effectuer une mise en relation arithmétique des données. L'opération numérique peut ainsi prendre place dans une écriture formelle du problème (type équation)

2.3. Description du logiciel

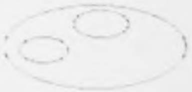
2.3.1 Contenu

Le logiciel SIPOS tourne sur Macintosh Plus. Pour une description plus complete voir ESCARABAJAL, 1988. Tous les echanges avec la machine se font par sélection sur l'écran, le clavier n'intervient pas.

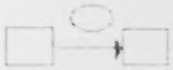
SIPOS propose trois modules OPERATION, SCHEMA, SOLUTION. Le premier est indépendant, les deux autres se succedent pour chaque problème traite.

Dans le module OPERATION l'enfant doit trouver quatre additions puis quatre soustractions que l'on peut poser avec deux nombres. Il dispose pour cela de deux nombres et d'un symbole d'inconnue "?" qu'il doit placer dans un gabarit d'operation (□ + □ = □). Le but de cette activite est d'entrainer l'enfant à placer l'inconnue aux differentes places de l'équation, et ainsi d'enrichir et assouplir l'écriture numérique d'operations pour la phase suivante, en particulier le module SOLUTION.

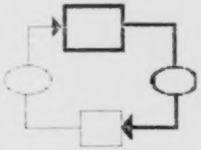
Dans le module SCHEMA, l'enfant doit, après la lecture de l'énoncé, choisir parmi les trois schemas proposés celui qui convient pour illustrer la relation



Parties-tout



Transformation



Comparaison

sous-jacente à l'enonce, l'instancier, c'est à dire le remplir en positionnant les donnees de l'énonce, nombres et inconnue, aux differentes places qu'offre le schéma choisi. Si l'enfant a commis des erreurs, dans le choix du schema ou dans son instanciation, le système envoie une page-correction expliquant comment il fallait "comprendre le problème", en proposant l'instanciation correcte du schéma.

Module SOLUTION : a la suite de la phase precedente, l'enfant doit poser l'operation permettant de résoudre le probleme. Une version miniature de l'instanciation correcte du schéma demeure sur l'ecran, dans le but de souligner la filiation entre le schema instancie et l'ecriture de l'operation. Si l'operation

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proposée ne permet pas de résoudre le problème, le système envoie une page-correction qui donne comme solution la mise en équation directement inférable du schéma.

2.3.2 Modalités de déroulement.

Dans la version testée, le logiciel propose 22 problèmes : 6 Parties/Tout, 8 Transformation et 8 Comparaison. L'ordre de présentation des problèmes est préfixé. L'exploitation complète du logiciel, c'est à dire la résolution correcte des 22 problèmes, demande plusieurs séances. Cette exploitation peut durer de 3 à 6 heures. Le système enregistre un historique du sujet lui permettant d'éliminer des présentations ultérieures les problèmes correctement résolus, et de reproposer à chaque nouvelle séance les problèmes incorrectement résolus.

3. LA VALIDATION

3.1. Objet de la validation

Nous cherchons à valider la pertinence d'un enseignement de la démarche de compréhension et de résolution telle qu'elle est proposée par le logiciel, à savoir : l'identification de la relation et sa traduction en écriture numérique. Nous entendons donc d'abord établir la faisabilité de l'enseignement de cette démarche pou l'âge considéré, et puis éprouver l'effet qu'il exerce sur la résolution de problèmes additifs hors de l'environnement du logiciel.

3.2. Les sujets.

Les sujets sont les 39 élèves de deux classes entières de CE2, école des Lilas (20) et celle d'Epinay (19). Dans chaque classe, ils furent répartis en deux groupes, expérimental et contrôle, de niveau équivalent, selon l'avis des enseignants.

3.3. La mise en place de la validation.

3.3.1. Pré-test et Post-test

Tous les sujets passent deux fois une épreuve-test, avant entraînement puis après. Le test comprend 11 problèmes : 1 problème Parties-tout, 4 problèmes de Transformation, et 6 problèmes de Comparaison. Nous avons éliminé les problèmes connus pour être maîtrisés en CE1.

3.3.2. Les activités du groupe expérimental

Une présentation collective des catégories de problèmes et des schémas les illustrant est d'abord faite en classe. Puis l'entraînement sur le logiciel se fait en passation individuelle, chaque séance durant de 20 à 30 mn. Au cours de la première séance le sujet aborde le module OPERATION (cf 2.3.1) il n'y revient plus ensuite. Six séances suivent, consacrées à l'entraînement à la résolution proposé dans les modules SCHEMA et OPERATION. Ce nombre de six séances fut arbitrairement fixé pour assurer une homogénéité de la durée de passation.

3.3.3. Les activités du groupe contrôle

Afin de permettre la comparaison des deux groupes, il est nécessaire que le groupe contrôle ait lui aussi une activité de résolution de problème, mais sans l'entrainement de la démarche spécifique à SIPOS. Nous avons choisi de présenter les mêmes problèmes que ceux du groupe expérimental, dans le même environnement Macintosh, mais sans le module SCHEMA : pas d'identification de la relation, ni du schéma la représentant, pas d'instanciation, ni de construction de l'opération numérique par filiation avec la relation identifiée. Le sujet a uniquement pour tâche de résoudre le problème dans le module SOLUTION, c'est à dire de poser l'opération qui lui semble pertinente. Cette activité s'est déroulée sur quatre séances et non six, le traitement de chaque problème étant, dans ce groupe, réduit par rapport au groupe expérimental.

3.3.4. Epreuve à long terme

Les deux groupes ont été soumis, quatre semaines après la fin de l'entrainement, à une épreuve à long terme différente pour chacun des deux groupes. Les onze problèmes de l'épreuve-test sont de nouveau présentés en passation collective, mais la présentation reproduit celle des séances d'entrainement. Pour le groupe expérimental, chaque énoncé est suivi, d'une demande d'identification et d'instanciation de schéma, et d'écriture de l'opération. Pour le groupe contrôle, seul ce dernier élément est demandé.

3.4. La mesure de la validation et ses indices

3.4.1. Pré-test/post-test

Cinq indices codent la réponse à un problème :

- Global : réponse verbale et résultat correct, 2 pts; résultat seul, 1 pt; échec ou Non Réponse, 0 pt
- Compréhension : pas de glissement de sens 2 pts; glissement de sens limité, 1 pt; glissement de sens important ou NR, 0 pt
- Numérique : opération correctement exécutée, 2 pts; erreur de syntaxe, 1 pt; opération incompréhensible ou NR, 0 pt
- Cohérence : lien entre opération et réponse verbale, 1 pt; pas de lien, 0 pt
- Nombre de soustractions : décompte des soustractions posées

3.4.2. Epreuve à long terme

Pour le groupe expérimental, quatre indices codent les réponses :

- Schéma : identification correcte du schéma, 1 pt
- Instanciation : instanciation correcte, 1 pt

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- Opération : l'opération posée permet la résolution du problème, 1 pt. La mise en équation est telle que la valeur de l'inconnue "?" sera effectivement la valeur attendue (cf. 2.3.1).
- Global : schéma instancié et opération correcte, 2 pts, un seul de ces éléments correct, 1 pt, aucun, 0 pt. Cet indice est construit de manière à être similaire à l'indice Global des épreuves tests.

Pour le groupe contrôle, seul l'indice Opération peut être attribué.

3.5. Les Résultats

Les résultats suivant sont partiels et concernent l'école des Lilas, la passation étant à ce jour en cours à l'école d'Epinay. Les tests de signification sur la comparaison des groupes expérimental et contrôle seront effectués après recueil complet des résultats.

TABLEAU : Resultats des deux groupes aux épreuves-test

		indice Global max 220	indice Compréhension max 220	indice Numérique max 220	indice Cohérence max 110	indice Nb de Soustraction sur 11 Pb
EXPE	pré	87	106	173	84	32
	post	110	134	190	103	48
CONT	pré	83	102	154	59	29
	post	100	108	171	62	36

Les performances des deux groupes (indice Global) au pré-test sont équivalentes et assez faibles (cf. tableau). La dispersion inter-individuelle est élevée (de 3 à 19 pts en groupe expérimental et de 6 à 21 en groupe contrôle). Nous démarrons donc l'apprentissage avec deux groupes homogènes présentant une grande variabilité quant aux performances initiales des sujets.

Pour le post-test, on observe une légère progression des deux groupes à l'indice Global (expérimental +23, contrôle +17). Cette similitude de progression se nuance par l'examen des autres indices. Pour l'indice Numérique, le gain (+17) est identique dans les deux groupes. Cependant, pour les indices Compréhension et Cohérence, les gains sont plus importants dans le groupe expérimental (+28 et +19 contre +6 et +3). Enfin le nombre de soustraction est lui aussi en progression dans le groupe expérimental (+16 contre +7).

Notons que les 23 points de progression du groupe expérimental (indice Global) sont équitablement répartis sur tous les sujets, tandis que les 17 points du groupe contrôle sont quasiment attribuables à un seul sujet.

Ainsi la comparaison [illegible] faibles dans les deux [illegible]

L'épreuve à long terme témoigne aussi d'une légère différence entre les deux groupes : 74 en expérimental contre 59 en contrôle pour l'indice Opération, qui est le seul indice comparable. En revanche cette épreuve montre une très nette progression de groupe expérimental (indice Global 130) par rapport au pré-test (indice Global 87). Mais ce qu'il est important de souligner pour ce groupe, est le fait suivant : dans 60% des cas (10 sujets x 11 problèmes) le schéma correct est identifié, et dans 56% des cas d'identification correcte, le schéma est bien instancié. Il y a manifestement eu apprentissage des relations lors de l'expérimentation sur le logiciel, mais sans transfert notable à l'épreuve de post-test.

4. DISCUSSION

En ce qui concerne la faisabilité de l'enseignement des relations additives en CE2, la réponse est oui , attestée par l'intérêt des élèves lors de l'exposé des schémas au groupe expérimental, leur participation active et leurs progrès aux séances d'entraînement sur le logiciel, ainsi que les performances évaluées par les résultats à l'épreuve à long terme.

S'il y a bien eu apprentissage, celui-ci ne se traduit que faiblement dans les épreuves post-test. Deux éléments de réponse peuvent être avancés pour expliquer ce fait. L'un tient à l'hétérogénéité de niveau des enfants, l'autre au poids des difficultés numériques dans la résolution de problèmes.

Lors de la construction de l'expérience de validation nous avons sous-estimé l'effet potentiel des variations individuelles. Ainsi, tous les enfants de niveau initial faible ou moyen n'ont pas atteint la fin du logiciel, pourtant conçu pour être exploité dans son intégralité. Au lieu de fixer un nombre de séances identique pour tout sujet, ce qui définit un critère de fin d'expérimentation, nous aurions pu fixer un nombre minimal de problèmes réussis, définissant ainsi un critère de fin d'apprentissage. Ce dernier point déborde la présente expérience et s'inscrit dans la problématique plus générale de la définition des critères d'évaluation d'un enseignement.

Dans notre conception du programme d'enseignement de SIPOS nous avons privilégié l'activité de compréhension de l'énoncé. L'accent fut donc mis sur l'identification de la relation et sur sa transformation en écriture numérique. En revanche, nous ne nous sommes pas intéressées à l'exécution du calcul. L'origine de l'échec du transfert à la situation de test réside peut-être là.

Durant l'entrainement, les enfants ont appris à se détacher de l'ordre de présentation des données dans l'énoncé, comme de l'ordre habituel d'écriture des opérations. Cet acquis se traduit par une apparition massive d'écriture horizontale (vs. en colonne) et, parmi ces dernières, une proportion importante d'additions à trou. La grandeur des nombres considérés fait que les procédures de comptage disponibles sont inapplicables et que seuls les enfants pouvant poser et exécuter la soustraction réussissent. Ce qui manque aux autres, et que notre logiciel n'a absolument pas pris en charge, est la pratique de la conversion de l'addition en soustraction. On peut envisager un troisième module au logiciel assurant l'entrainement à cette pratique.

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HOW BIG IS AN INFINITE SET? EXPLORATION OF CHILDREN'S IDEAS.

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Children were required to compare the size of different pairs of sets. In particular, the set of all numbers was compared to a finite set considered very large by the child. They also located symbols of different sets on a long strip in an attempt both to order them according to size and to represent the gaps between them by the distances. Although children (from about 8 on) generally knew that the set of all numbers was the largest, the infinite gap between this set and huge finite sets was not conceived until much later.

> Above everything, we must realize that "very big" and "infinite" are entirely different. ...There is no point where the very big starts to merge into the infinite. You may write a number as big as you please; it will be no nearer the infinite than the number 1 or the number 7.
> Kasner and Newman, Mathematics and the imagination (1949, p.34).

One of our colleagues, a psychologist, said recently in a lecture: "Now, that we know how to do it, we can run an almost infinite number of experiments." Although this particular bright young man is well aware of the difference between a big finite set and an infinite set, we quote his statement as one example out of many utterances in daily discourse, in literature and in the media, where "infinite" is used to designate large but finite. We suspect that such a

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prevalent figure of speech, although used metaphorically, occasionally reflects some ambiguity in adults' minds and helps to blur children's distinction between very many and infinitely many.

The aim of the present research is to explore children's ability to distinguish between finite sets that they consider very big and the "smallest" infinite set, i.e., the set of natural numbers.

Comparison of Sets

We presented 38 children, aged 5 to 13, with simple questions requiring a judgment of "what are there more of, X or Y?", where X and Y were replaced, in turn, by members of different sets, like leaves on all the trees in the forests, hairs on all people's heads, grains of sand on earth, etc. The set that was judged larger in each step was next compared to a new one. The "winning" set in that sequence of binary comparisons (assuming transitivity) was then compared, in our target question, to the set of all numbers. The children were also asked to justify their judgments.

The percent of children who knew that there are more numbers than members in their largest finite set, and knew why, in each of three age groups, was as follows:

5-7: 29%; 8-10: 72%; 11-12: 83%

Here are two examples of good explanations:
Mor, age 8: "There are more numbers, they never end. Grains of sand will take you perhaps two thousand years to count, but at some point we'll finish them, they will end up for you." Eitan, 9(3): "Numbers, because the numbers go on forever, and grains of sand ... there are, say, on earth million grains of sand, then the numbers can go on: million and one, million and two, and so on without end."

The following are examples of children who scored negatively:
Omri, 5(10): "There are more leaves, because on each tree there are many leaves, and numbers there are more fewer." Daniel, 7(4): "Sand, because there is no number more than tri-ion and there is plenty of sand in the world ... wherever you go you see sand."

Location of Sets

In the course of our research on potential infinity in children (Falk et al., 1986), it became evident that sometimes children were able to say that "there is no end to numbers," but they were unable to apply the implications of that statement beneficially in a competitive game. Children had often heard from parents or siblings that there are infinitely many numbers, without internalizing the meaning of this contention. We therefore wished to test the understanding that not only are there more numbers than grains of sand, but that there are infinitely more numbers than objects in any finite set. Consequently, we asked the children also to represent the gaps between different sets. This exploratory task was run with 91 children (48 boys; 43 girls) aged 6-12.

Method. Small cards, each bearing the name of one of the compared sets, were prepared. They were presented after being through with Comparison of Sets. We read together the names of the sets, then we asked the child to arrange the cards from left to right according to their estimated set size. A long strip (ca. 90cm) was now presented, and the child was instructed to carefully paste the cards on the strip so that the spaces between the cards would represent the gaps in the set sizes. Before doing the job, the procedure was explicitly demonstrated and explained with familiar sets like "fingers on one hand" "fingers on two hands" and "all the people of Jerusalem." The children were guided to keep the two first cards very close to each other so as to leave space for the much bigger difference between the second and third set. We tried to explain the task of representing the gaps proportionally by the distances, without using technical terms. Children were also asked to explain their placements.

We offered this task halfheartedly, since the correct response to the task of placing the set of all numbers should be to refuse to do it. No finite strip, whatever its length, can suffice for placement of that card, no matter what scale is applied. Presenting an impossible task to children, who are eager to cooperate and comply with the experimenter's requests, is not a desirable procedure. We felt, however, that if we do only with Comparison of Sets, we run the risk of missing important information. Consequently, one of the most important features of the procedure was our last instruction: "If you feel the strip is not long enough for you, please let us know, and we'll see what we can do about it." If a child did respond that way, we tried to find out why did the strip not suffice. We asked whether an extension would solve the problem, how long an extension, and if not, why.

Analysis of results. Let m(set) denote the measurement in centimeters (reading from the left end of the strip) of the location of (the middle of) the set's card. A quantitative index that we call Critical Ratio, CR, was extracted from the production of each child:

$$CR = \frac{m(\text{All Numbers}) - m(\text{largest finite set})}{m(\text{largest finite set}) - m(\text{smallest finite set})}$$

CR assumed a *negative* value whenever some finite set was estimated to exceed the set of all numbers. When a child refused to place All Numbers, even though we offered as long an extension of the strip as she wished, we assigned a $CR=\infty$ to that performance. Determination of the status of a child's understanding was not as clear-cut for positive and finite CRs. We further differentiated between $0<CR\leq 1$ and $CR>1$. The latter signifies at least an understanding that the difference between All Numbers and the largest finite set is greater than the gap between any two finite sets.

Observing the children, whose CR was positive and finite, at work, and listening to many of them think aloud, we realized that in most cases they were not comparing the distance between All Numbers and the largest finite set to that between the largest and smallest finite sets. They were comparing it, instead, to the distance between the largest finite set and the one before it. Apparently, children could not cope with simultaneously comparing all the distances they were creating to each other, and they were focusing on two at a time. Consequently, we decided to modify our index so that it would better reflect the subjects' decision process. The revised measure was

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$$CR' = \frac{m(\text{All Numbers}) - m(\text{largest finite set})}{m(\text{largest finite set}) - m(\text{one before largest finite set})}$$

Whenever CR is negative (or zero) so is CR', and the same is true for an infinite CR. For positive finite values, $CR' \geq CR$.

The results are presented in Table 1. Children are distributed according to years of age. In each age (row), they are distributed in percentages into four levels of performance. Besides the children's measure of performance, we also considered their verbal responses. Children who unequivocally expressed an understanding of the infinite distance between the largest finite set and All Numbers were noted. The percent of these children (out of all children in their age group) is given in the last column of the table.

Table 1 shows a decline with age of negative CRs. The percentages of higher levels of performance are increasing with age only very globally. On the whole, the slope of the developmental progress is very shallow. The same is true with respect to the mean ages of children in the different levels, presented in the bottom line of the table. The means have to be considered relative to 9(4), i.e., the mean age of all 91 children. These results can partly be accounted for by the rather small sample in each year of age. They may also reflect a real high variability within age groups, and a weak polarization with age, of a concept that is not learned in school. The index CR might possibly be inappropriate. This experiment is now going through extended replication and reexamination of procedure and analysis.

Verbal responses. Children's explanations were often informative. Let us consider a few examples.

The first, illustrates misunderstanding of both the infinity of numbers and the infinite gap between them and finite sets. Carmit, aged 10, placed "leaves on all the trees", "grains of sand on earth" and "all numbers" adjacent to each other in increasing order. "Grains of sand and numbers have about the same number, this is the biggest number that both you and I don't know, but it exists, therefore I cannot show the difference between them by a space."

The following, would probably be positively scored were

Table 1. Distribution of children by age and by level of performance (in percentages).

Age	Level of performance				Total (absolute number)	Verbal under-standing
	$CR \leq 0$	$0 < CR' \leq 1$	$CR' > 1$	$CR = \infty$		
6	83	8	8	-	99 (12)	-
7	40	7	20	33	100 (15)	20
8	21	14	64	-	99 (14)	-
9	25	38	38	-	101 (8)	-
10	18	29	35	18	100 (17)	18
11	20	13	33	33	99 (15)	40
12	10	20	40	30	100 (10)	30
Total (absol. number)	31 (28)	18 (16)	34 (31)	18 (16)	101 (91)	16 (15)
Mean age	8(3)	9(10)	9(8)	10(1)	9(4)	10(7)

we to test them only in Comparison of Sets. They did say that there is no end to numbers, but betrayed their lack of assimilation of this contention by some additional comment or act. Eyal, 8(6), commented while locating All Numbers not at the extreme right end of the strip: "There is an infinite number, so I put them almost at the end" (italics added). Yair, 8(6), placed leaves, sand and numbers in increasing order, at the edge, very close to each other: "There are more numbers. Even though there are plenty of leaves and grains, they end, and numbers do not end. There are more numbers, somewhat more but not a lot more."

Several children, not only refused to place All Numbers, but justified it in a near perfect way so as to leave no doubt concerning their conception: Einat, 10(10): "There is no place for numbers. There are numbers up to infinity and a fixed amount of grains of sand, so the distance between them is endless, it is therefore best not to put it at all." Yosi, 11(4): "I have no place for numbers because there is no end to the numbers in the world. You can place on the strip only things that have an end." 'Which length of a strip do you need in order to be able to locate All Numbers?' "There is no such a strip ... I can take a strip from here to America and it would not suffice, even if I'll put a strip on all the length of earth there would be no place because there is no end to numbers." Itay, 11(5), refused to paste both Grains of Sand and All Numbers: "The space is not sufficient for the last two sets. There is a giant gap between the leaves and the sand, it is a giant distance, it will take a lot of time to count, but there is a gap. However, with respect to All Numbers, it is impossible to count, so it is impossible to determine a distance. I cannot represent All Numbers."

Discussion

There is no gradual transition from large numbers to infinity. Conceptually, a qualitative discontinuous leap is required. Still, the conjunction of the results of Comparison and Location of Sets suggests that, psychologically, the way to infinity is not that simple. Some gross developmental stages seem to shape up.

Most of the youngest children think that the set of all numbers is a large finite one, not necessarily greater than distinctively big finite sets. This belief decreases steeply with age. From age 8 on, only a minority of the children

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think so, and they often convincingly verbalize the claim that there is no end to numbers. We could have thought that they fully understand the infinity of numbers, if not for their responses to Location of Sets. Throughout a long period, extending to age 12, or possibly beyond it, the majority of the children are content with representing the distance between a huge finite set and All Numbers on some finite scale. Among 11 and 12 year olds, only 30-40 percent insisted on the impossibility of that task.

Admittedly, Location of Sets is loaded with problems. In particular, we sometimes cannot know the status of understanding of children who have placed All Numbers at the right-hand end, far beyond all finite sets. Possibly, they do grasp the infinite nature of the gap, but they have represented the gaps only ordinally, or they tried to conform to the instructions of locating all cards. Still, in many cases, children's beliefs could be diagnosed by that task. Certainly, there was no doubt concerning children's understanding in the extreme cases of either a negative or an infinite CR. The hardest to interpret was a performance that resulted in a finite positive CR. If, however, the child located All Numbers not at the extreme right end of the strip, leaving extra unused space (as did 17 subjects) it was indicative of lack of conflict between the finiteness of the strip and the unending gap to All Numbers. Also, when All Numbers and Grains of Sand were placed adjacently (as did 8 subjects), it could be interpreted to reflect the belief that there are almost as many grains of sand as numbers. Finally, some of the better explanations of the abysmal gap to All Numbers, cited above, seem to reflect a level of full understanding, to the extent that any of us can ever completely comprehend infinity.

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HYPOTHETICAL REASONING IN THE RESOLUTION OF APPLIED MATHEMATICAL PROBLEMS AT THE AGE OF 8-10

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Summary

I have observed a wide use of hypothetical reasoning in the resolution of applied mathematical problems by children from the age of 8 to the age of 10 who are testing our project. Such use seems to be related to problem solving skills, even if problem solving seems not to be the most suitable setting in order to develop hypothetical reasoning. I give various examples of such pattern of reasoning and try to analyze its functions in problem solving activities. I present also some settings which can force a correct use of hypothetical reasoning and point out some open problems.

1. Introduction

In the context of the project for teaching mathematics in primary school, to which are related various researches of our group's on problem solving in curricular environment, we have noticed some interesting phenomena with regard to the presence of hypothetical reasonings in resolution procedures, as a wide use and a good mastery of it by good problem-solvers or also a quite wide presence in some kinds of problems and nearly total lack in others.

So it seemed to me useful to focus the question of hypothetical reasoning in problem solving environment, to explore its functions (section 4) and to find out some conditions which can make easier or hinder the proper use by pupils (section 5). This is the beginning of a research, not an accomplished one. Till now my work has concerned only the analysis of hypothetical reasoning as expressed by children and the statement of some related hypotheses.

I have chosen to observe children at the age of 8-10 because it is the period during which 'our' pupils are widely requested to write down their own solving strategies and because I think it is a crucial age as to the development of problem solving skills, for at that age children usually acquire attitudes, behaviours and concepts which strongly condition their problem solving performances afterwards.

As 'applied mathematical problems' I mean all those problems that concern aspects of the real world (occupying or dividing portions of the classroom, planning the utilization of periods of time, estimating of expenses for travels or purchases, projecting scientific 'experiments'...) and need the use of mathematical concepts and skills (in arithmetical or geometrical setting). I have chosen to refer exclusively to these problems because they are particularly frequent (at the age of 8-10) in our project for primary school and the behaviours I am interested to analyze in this report appear very frequently in some of them.

2. The context and some related questions

As previously mentioned, I am going to refer almost exclusively to pupils who are testing our project for primary school. As regards the purposes of this research, I deem these features of our project the most relevant (see [Boero, 1988]):

- applied mathematical activities are the privilegiate setting for the construction of mathematical concepts and procedures;

- teachers are engaged on teaching language competences, mainly with the purpose of the production of spoken and written texts, in suitable settings to force a syntactical enrichment (descriptions of productive processes, of the behaviour of relatively complex machines, etc), since the beginning of primary school. Among the activities performed which such purposes, I mention: children's verbalizations of their own strategies, which is requested since the age of 7 so that they can account the reasoning they have used in the resolution of the problem; the resolution of word problems without explicitly given numerical data or geometrical problems; the construction and verbalization of computational strategies in applied arithmetical problems with numerical data, before the introduction of written computation algorithms.

As to the experimental conditions in which the protocols I have taken into account in this research have been gathered, they are the same already stated in [Boero, 1988]; particularly, the major part of the records we have examined come from 'observation classes', which we look after with particular care and more widely from the first year to the fifth (always with the same teacher(s), as customary in Italy), in order to get more detailed and complete information on both the whole work and the features of the pupils involved.

3. Verbal manifestations of hypothetical reasoning: typical situations

The verbal manifestations of hypothetical reasoning we consider in this section can be viewed as 'locally' spontaneous, in the sense that they have not been preceded by discussions nor by teacher's interventions addressing to such pattern of reasoning; nevertheless, it is necessary to point out the large extension of verbalization activities in the whole project, and also that the best written pupils' texts describing their own strategies (when effective) are strongly emphasized by teachers when pupils are comparing their strategies each other.

We have observed three main kinds of hypothetical reasoning in applied mathematical problem solving.

(A) Hypothetical reasoning related to problems with alternative initial data or conditions which may change the resolutive procedure. In some cases (A1) the alternatives are explicated in the task given by the teacher or agreed in class, in other ones (A2) the pupil himself must single out such alternatives. Examples of A1 settings are the extimates of expenses for travels, depending on the means of transport chosen (at the age of 10), or for productions to be made in classroom (cakes, biscuits, ...), depending on the amount of product one want to get (at the age of 8-9).

Let us consider the following example (example 1): the children are requested to calculate the cost of a pizza - ingredients and tools - if it is produced once and if it is produced twice. Reasonings as the following are frequently carried out by children related to such problems:

"If we want to make the pizza for all the class we have to buy ... and then we spend totally...; if we want to make the pizza twice we should spend a double amount for ingredients, but nothing more for tools, for we have already bought them, ..."

(B) Hypothetical reasoning related to the alternatives or difficulties one may find when executing a procedure, or conditions which the solution must fulfil. In these cases pupils have to follow (B1) or state in a utterly autonomous way (B2) alternative working hypotheses related to such conditions. Among the situations B1 hypothetical reasoning is often found when in order to buy some goods agreed in class an extimate of expenses is made which must be related to the amount of available money (example 2):

"If we buy, ...we spend... But we have got only... and so we must reduce the amount ... not to exceed the money we have..."

Examples of situations of kind B2 in our project are frequent, at the age of 8-10, as when pupils autonomously make hypotheses on how to divide a space in a given number of equal portions (for example, to represent on a wall of the classroom a given historical period) and show how they must act if such hypothesis does not lead to a correct solution. In the following example (example 3), the task is to represent a 54 years interval (1935-1988) by joining some sheets on a wall so that it is completely covered. Francesco is a child coming from socially disadvantaged background but very good problem solver.

Francesco: *"We have already drawn the 'line of time'. You have to take a large paper-sheet, say 25 cm long. With a ruler you measure the sheet. It must be 25 cm long, if it is longer, make a bar at 25 cm, if, on the contrary, it is shorter, join another sheet. The sheets must be totally 54. Then write the years: 1935, 1936 and arrive till at 1988. But if it is larger (than the wall), you try with a lower measure, or, if you have spared some space on the wall, you join equal parts to the base, longer, ..."*.

Teacher: *"I do not understand"*.

Francesco: *"If you have spared some space, you measure it and divide into 54 equal parts, and join one part to each sheet"*.

Another situation in which pupils have used widely hypotetical reasoning is the following (example 4): this year the class has been transferred to another building. The 'new school' is very close to the 'old school'. If pupils want to go on measuring length and directions of the shadows of a nail, as they had done last year, a question arises if they must go back to the old school's garden or they can perform measurements in the new one, without affecting them. An experiment is planned: a group of pupils goes to the old garden, another group to the new one. The two groups perform the measurations exactly at the same time, in order to check whether the measures are equal. Last year they had been measuring the shadow of a nail driven in a plywood board and sticking out of the board by 4 cm. Now they decide to construct another board (with polystyrene as they have no more plywood) and drive in it another nail. The length of the portion of the nail sticking out of the board must be the same as the other nail. The new and the old board must be each in a horizontal position, one in the new garden, one in the old. These conditions have been suggested by pupils, with some assistance by the teacher. In the protocols we have found a lot of reasonings as the following:

" ... *take a nail 6 cm long; now measure the thickness of the new board. If it is not the same as the plywood board, do not drive it in the board by more than 2 cm. So you have left 4 cm sticking out of the board, which is the true length of the nail and is the same as the plywood board. .... Go to the garden and put the board in the sun, in the same position as last year. If it happen that the board is lying on the gravel, then level the ground until it becomes a smooth plane. If you cannot go on levelling it and there are still some stones you cannot remove and make the board swing, it is unpleasant, because shadows are not comparable. Then lay the board on the pavement (which is very close) in the same direction as last year. ...*".

In this case the function of hypothetical reasoning is foreseeing the situations which may affect the soundness of the whole procedure and planning suitable interventions.

(C) Hypothetical reasoning is also present when children are comparing different strategies (age of 9-10); sometimes, in our classes, children, after they have solved a problem individually, are requested to compare their strategies each other; then we can observe kinds of hypothetical reasonings as the following (example 5):

"I have done in this way... if I had done as Claudia I should have found:... Then the reasoning of Claudia needs longer and more difficult calculations than mine..."

The list of situations in which the use of hypothetical reasoning is shown is related to what we have observed up to now. There may be other situations in which hypothetical reasoning can be found. I want also to specify that in a great number of situations we can find different kinds of hypothetical reasoning (in particular, A and B) at the same time, as in examples 2 or 4. On the other hand, our classification itself must not be taken as a rigorous one, as there is not a very sharp boundary mark between situations of kind A and of kind B.

In all the cases we have taken into account, we have found that hypothetical reasoning is developed in a correct way mainly by average and high-level pupils; in situations of the kinds A and B children using more widely and properly hypothetical reasoning are good problem solvers, and, conversely, good problem solvers are usually good performers of hypothetical reasoning in situations A and B.

I remark that the use of hypothetical reasoning in other settings is not easily transferred to applied mathematical problems. It seems that who uses properly hypothetical reasoning in problem solving is already able to use it properly in other settings. This can strengthen the hypothesis that problem solving activities are not very suitable to force the acquisition of hypothetical reasoning, but they are a good opportunity to refine, extend and develop it.

4. Functions of hypothetical reasoning in the resolution of applied mathematical problems

The following remarks about what described in the preceding section can be interesting as to the didactical choices to be performed to improve problem-solving skills of pupils and, more generally, to improve mathematical instruction.

Hypothetical reasoning plays seemingly different functions depending on the different cases we have considered.

In situations of kind A it is useful for children to become conscious (in a more or less autonomous way) of the situation and to find the resolution procedure, with suitable adjustments to initial conditions. The capability to get on taking into account the different conditions is related to deductive reasoning, as far as it can be performed in primary school. It is widely diffused the opinion that at this age deduction cannot be viewed as the application of rules (see, for instance,[Johnson-Laird, 1975]). My research sketches a possible way to overcome the difficulties widely supported by evidence in developing reasoning skills (see, for instance, [O' Brien and others, 1971])

In situations of kind B hypothetical reasoning can play two functions very important and strictly related to the construction of the resolution strategy, which explains why there is a good correlation with problem solving skills:

- a central role of support to the planning of the resolutive strategy: the pupil is forced to analyze his procedure and foresee the arising of difficulties or alternatives. He can put himself in a particular case (stating some particular hypothesis) and then generalize it with suitable adjustments if it does not fulfil the conditions previously stated.

This is a heuristic process which we have found mainly in problems related to measure or geometrical constructions (see [Ferrero and Scali, 1987]). It is transferred to a great deal of children as a result of the construction of computation strategies to get numerical results before the introduction of standard written division algorithms. For example Francesco, the pupil of example 3, few months later, searching for the amount to be paid by each of the 19 pupils of his class for a trip whose total cost is 74'000 lire, writes (example 6):

"If any child pays 1'000 lire it makes 19'000; then let us try with 2'000 lire: it makes 38.000. It does not yet suffice. If any child pay 10'000 lire, it makes 190'000, which is too much, ..." (and so on, by trial and error, he comes to "little less than 4'000 lire" and then, specifying *"how much one has to take off"* he comes to *"about 3'900 lire"*).

- A more restricted, but not negligible role of plain adjustment to the conditions which have been posed on the solution: in this case (see example 2) we have already remarked that the working hypotheses are not chosen by the pupil but are already given by the setting in which the problem is stated.

In situations of kind C hypothetical reasoning is used by pupils to reflect on their own strategies; they have to evaluate the consequences of the choice of alternative procedures. This capability, according to our observations of pupils, seems not strongly correlated with problem solving skills, as it can be found even in pupils not very clever in planning. Nevertheless, it is relevant for the mathematical instruction, as it plays an important role in a great deal of mathematical activities (analysis of algorithms, checking of proofs, ...).

Generally speaking, it can be remarked that hypothetical reasoning in applied mathematical problem solving is relevant, in different ways, in order to construct correct and fruitful attitudes (analysis of conditions, discovery of consistent strategies, checking of the solutions, statement and investigation of resolution hypotheses, ...), which are also related to other mathematical activities (proofs, ...).

5. Conditions for the development of hypothetical reasoning in applied mathematical problem solving

The purpose of developing hypothetical reasoning (in the different forms and with the different functions mentioned in the previous section) seems very important related to mathematical instruction. So it is is interesting to ask what conditions may make it easier. Based on what remarked up to now, it seems to me that it is necessary distinguish between conditions external to resolution processes and conditions internal to it.

As already remarked in section 3, problem solving activity seems not to be very suitable to construct the linguistic and logical tools that are needed to develop hypothetical reasoning. After a long term (2-4 years) analysis of each pupil's performances, I think I can conclude that pupils, to a great extent, improve their use of hypothetical reasoning in mathematical problem solving only if they have already achieved a first-level mastery of it in extra-mathematical environments .

Moreover, I think that the development of hypothetical reasoning inside applied problem solving situations needs some careful choices by teachers regarding the following aspects.

- The choice of the context for problem solving activities: problems which are viewed by children as artificial or not very well related to their experiences and concerns often lead to stereotyped answers (if they are straightforward) or to blockages (if they are difficult); in such cases the pupil hardly engages himself in the endeavour of stating working hypotheses or of carefully analyzing data and conditions which are inherent to the situation. Very often the pupils, even if he can find a solution, does not compare it in a critical way with the situation. On the other hand, the relevance of context in problem solving has been very well pointed out by a great amount of research (see, for instance, [Lesh, 1981, 1985]).

- The choice of the problem solving situations and of related tasks (either they are stated by the teacher or discussed and agreed with pupils): as previously mentioned, hypothetical reasoning can be found widely only related to particular features of the situation or to precise conditions posed by the teacher or to requests to perform some activity. For example, hypothetical reasoning hardly can be found when the task is to solve a problem which needs only recalling and applying a resolution procedure the pupil has already learnt and used in analogous situations, or when the resolution procedure is constructed by a sequence of steps strictly related each other without the arising of alternatives or difficulties, or when the numerical data distract the pupils from reflecting on the procedure. I have also observed that conditional sentences are used rather widely by pupils when planning the resolution of a problem, whereas they are hardly used when recording the resolution they have already found. It seems to me that in the former case pupils are interested in relations between hypotheses and consequences, whereas in the latter they are more interested in the sequential structure of facts. This may be related to the well known difficulties of students in understanding implication and other logical schemes (see, for instance, [O'Brien and others, 1971]). It seems that 'open' situations, in which reality is orienting and stimulating pupils, are the most suitable to stimulate heuristic activities and hypothetical thinking.

- The interventions of the teacher during pupils' work; let us consider the following protocol (the task is the same as in example 3)

Sabrina: *"In my opinion it can be performed in this way: you take a large paper-sheet, put it on the ground and count from 1935 to 1988, and they are 54 years. Then mark a space 25 cm long with a 50 cm long ruler, and draw a line with the pencil; if it is exact, you can trace over with a marking pen. Hang it on the wall and in each interval write the year."*

Teacher: *"And if it is not exact?"*

Sabrina: *"If 25 cm are too much, try taking away 5 cm, and it leaves 20 cm; try performing it 54 times, and if it is longer than the wall, it means that it is still too much. Then take away more centimeters."*

Teacher: *"But it may happen that, after much taking away, it becomes too short. Explain this other hypothesis."*

Sabrina: *"If after much taking away the sheet becomes too short, join a little squares, so that they are enough to get the exact measure."*

It is evident that teacher's intervention, relying on Sabrina's already constructed competences in verbalizing hypothetical reasonings, helps her to develop such reasoning according to the situation. In effect, after few months the protocols we have examined show that Sabrina has nearly achieved a remarkable self-confidence and autonomy even in more difficult problem solving situations than the preceding one.

The comparison between the resolution strategies chosen by each pupil is an effective way to force the development of hypothetical reasoning in applied mathematical problem solving. For pupils who are 'close' (in the sense of [Vygotsky, 1975]) to a proper and autonomous use of hypothetical reasoning in problem solving environment, going over their friends' hypothetical reasonings, adjusting them and comparing them with their own (eventually only sketched) reasoning is decisive as to get substantial improvements in this matter.

6. Open questions

At this point of the research, in my opinion, there are three kinds of open questions.

- Refinement and checking of the hypotheses I have stated in this report (in particular: as regards the correlations among the various forms of hypothetical reasoning shown by children and the more general applied mathematical problem solving skills; as regards the assumption

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that linguistic general competences in verbalizing hypothetical reasonings must go before the use of such reasonings in problem solving; as regards the conditions which seems to make easier the development of hypothetical reasoning in problem solving environment.

- Comparative studies on the kinds of resolutive strategies that hypothetical reasoning (as verbally expressed) could make easier, referring from one hand to pupils who are going out of primary school, after a 5 years training with our project, on the other hand to pupils who are coming into our experimental classes in comprehensive school (age 11-13) from traditional instruction at primary school level. In effect, it can be interesting to know if the wide use of hypothetical reasoning we can found in our pupils, in suitable settings, (which is not very usual in italian primary school, at the same age and with the same kind of problems) is related to the insistence of 'our' teachers in requiring that pupil explicate, in verbal form, their thoughts, or to the acquisition of deeper an richer ways of thinking than the usual ones of chidren at the same age.

- Statement of further hypotheses, as regards the influence of reasoning skills developed in applied mathematical environment on problem solving skills related to internal questions of pure mathematics. It seem also interesting to explore the connections between the development of hypothetical reasoning in situations of kind B2 and the development of more general skills of autonomous planning in applied or pure mathematical or extra-mathematical environment.

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TWO DIFFERENT VIEWS OF FRACTIONS: FRACTIONATING AND OPERATING

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In this paper, a description of sources of failure, identified from the solution strategies used by students in problems corresponding to two different meanings of fraction of a unit, is presented.. To carry out the study a test was applied to 111 pupils (between 11 to 14 years old) in the first year of the secondary school. After the data obtained were analyzed, twelve of the students were interviewed individually.

Introduction

Building up rational number knowledge requires the identification of different meanings of that concept. Kieren, T., in 1976, identified seven interpretations of rational numbers. Since then, researchers throughout the world have redefined this classification or used other categories of constructs to carry out further investigations related to pupils' understanding of the different meanings of the aforesaid concept (see for example Behr, M.; Lesh, R., and Post, T., 1983).

From a different point of view, Frendenthal, H., in 1983, has highlighted not only different meanings of fractions but also diverse functional characteristics within the situations in which the fractions are embedded.

For several years, a group of Mexican researchers have been working on a research project in an attempt to better understand the learning processes involved with school mathematics. As part of that study, an investigation which traced the development of the concept of fraction of a unit, taking into account the different underlying meanings in various teaching models of fractions, was carried out. In this paper, a description of sources of failure, identified from the strategies used by pupils in problems corresponding to a) the primitive Egyptian and b) the discrete model of fractions, is presented. The complete report of this investigation can be found in Figueras, O., 1988.

Characterization of a teaching scheme; Teaching models

An analysis of the teaching strategies included in the compulsory Mexican textbooks for the elementary school was carried out. The aim of this analysis was to identify those meanings taken into account in the didactical approach and to characterize their organisational structure.

The didactical setting as a whole was characterized as a conceptual net connected by diverse teaching models. A teaching model was defined as the set formed by: 1) the specific meaning of rational number intended to be taught, 2) the treatment employed to teach that meaning, 3) the language used in the teaching strategies, 4) the necessary abilities required to understand the meaning via the treatment, and, 5) the inherent relationships between all those elements. (For further details see Figueras, O.; Filloy, E., and Valdemoros, M., 1987).

The organisation of the didactical approach, through a juxtaposition of teaching models, establishes links between different meanings of the concept of rational number. By establishing those links, conceptual chains are formed which lead to the construction of a conceptual net aimed towards more abstract significance. The ultimate goal of the teaching procedure is to provide the conception of a rational as a number.

In the teaching scheme for the elementary school various teaching models were indentified. Those considered for this study were characterized as follows:

The primitive Egyptian model. The meaning underlying this teaching model is that of a fraction of a unit. Fractions appear as a means of relating a part with the whole. The treatment used in our textbooks highlights the fraction as a "fracturer" of a continuous whole. This aspect of a fraction is generally illustrated by geometric figures or pictures of an object (for example, a biscuit or an orange), which are used for partition and splitting processes - the well known "pie model".

The discrete model. This teaching model comprises the meaning of a fraction of a unit for discrete wholes - a collection of objects. Fractions are seen as a means of relating a subset with a set. The meaning underlying this treatment is that of a fraction as a quotient. Since the emphasis is placed on the result of a sequence of actions, fractions emerge as an operator. This particular treatment includes problem solving situations, which quickly lead to computational exercises.

A tendency to use pictures to contextualise the concept of fractions was revealed by the analysis of the textbooks mentioned earlier. Furthermore, the role of drawings is fundamental in the primitive Egyptian model. Therefore, the abilities necessary to interpret and use the symbolic - geometric language included in the drawings become of primary importance. The design of the items used throughout the study took into account this aspect.

Methodology of the study

The investigation was approached in four stages. The first three of them will be outlined briefly, the reader is referred to Figueras, O.; Filloy, E., and Valdemoros, M., 1985, 1986, 1987 for more details.

First stage. An exploratory study in which children 11 to 13 years old were interviewed and video-recorded. The protocols included items designed after the most significant difficulties were identified in a comparative analysis of the tests used in two studies carried out in Mexico and the one used by the CSMS Project (see Hart, K., 1981).

Second stage. A diagnostic test, that enabled the investigation of several aspects of the concept of fraction related to the teaching models considered, was applied, at the beginning of the academic year 1984, to a group of 32 students in the first year of secondary school (children between 11 to 14 years old). A qualitative analysis of the data was carried out, which produced a classification scheme of the answers and the strategies employed by students to solve the items in the test.

Third stage. The diagnostic test was applied a second time, in 1985, to a group of 43 first year secondary students. The data were characterized using the classification scheme obtained in the preceding stage. The main purpose of repeating the observation under the same conditions was to identify those categories of strategies and difficulties that appeared regularly.

Fourth stage. A third observation was carried out in 1986 with a group of 32 students under the same conditions as before. The students' strategies and answers were classified using the same scheme.

A factorial analysis of correspondence (with the collaboration of Dr. Françoise Plunvinage) was carried out using the data obtained in the second and third stages of the study (79 students). The main purpose of this analysis was to choose a representative sample of students of the last group. Twelve pupils were chosen to be interviewed individually in order to investigate the difficulties they found in transferring their knowledge related to one of the meanings of fractions, to problematic situations linked with the other meaning considered.

Results and Discussion

The classification of the strategies employed by the pupils to solve the items of the diagnostic test comprised two main types: (a) those that led students to success and (b) those that led them to failure. The latter were deemed to be of primary importance for the design of alternative teaching strategies which might enable pupils to overcome the barriers encountered. It is these strategies which are discussed in this paper and are referred to as sources of failure.

Three different types of phenomena were identified (a) sources of failure that emerged in a solution of an item related either to the primitive Egyptian model or to the discrete model, (b) and (c) correspond to sources of failure which arose only in the solution of the items linked with one of the two teaching models.

For the first group four categories were identified. Three of them are related to fundamental aspects of the concept of fraction and have been denoted by: 'Neglecting the given whole', 'The predominance of the cardinality of the part' and 'The predominance of the denominator'. The fourth one, 'Counting errors' includes all those strategies in which errors in the counting processes students employed were found. Thirty per cent of the population committed at least one of these counting errors. (The biggest collection of objects pictured in the items contained 108 elements displayed in group arrays.) Even though this category is not of obvious significance in the understanding of fractions, it shows that elementary school mathematics is not providing children with a knowledge of counting strategies.

Neglecting the given whole. This group of strategies contains all the cases in which the denominator identified by the student differed from that linked with the whole represented by an image. Examples of answers classified in this category are shown in figure 1.

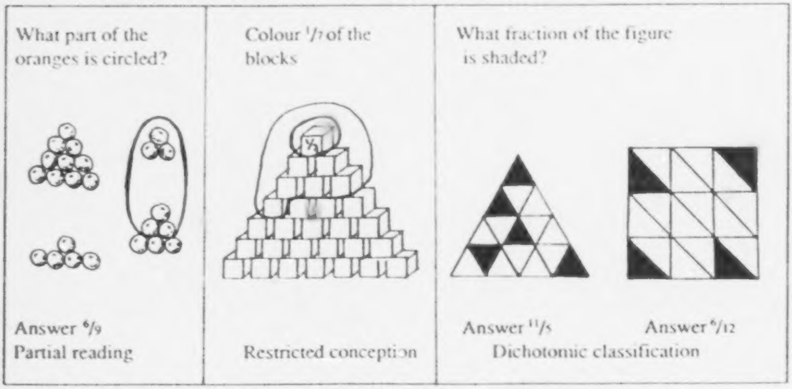

What part of the oranges is circled?
Answer $^6/_9$
Partial reading
Colour $^1/_7$ of the blocks
$\frac{1}{7}$
Restricted conception
What fraction of the figure is shaded?
Answer $^{11}/_5$
Answer $^6/_{12}$
Dichotomic classification

Figure 1: Examples of answers classified in 'Neglecting the given whole'.

This source of failure appeared, in various forms, more frequently in items corresponding to the discrete model. Two main problems were identified within the category encompassed by these phenomena. One of them is related to readability and the other is linked with specific meanings assigned by the students to the concept of fraction.

Readability is connected with difficulties of decodification of graphic language. These appeared with pictures having a specific features which induced a perceptual focus to a part of the picture that led students to a redefinition process of the given whole. In figure 1 an example of an answer classified in this subcategory is shown.

Two different misconceptions of the meaning of a fraction assigned by the students were identified; one was called dichotomic classification and the other is referred to as a restricted conception of fraction.

In the answers grouped in the dichotomic classification a particular quality used as a graphic code (as colour or shade) is employed by the pupil as a means of separating the whole into two disjoint sets: a set of elements with a certain property and its complement. The cardinality of the first set is associated with the denominator and the cardinality of the complement to the numerator. The examples in figure 1, given by the same student show that this classification can be made without responding to a specific pattern. The numerical expression given by the student does not express the part-whole relationship expected, instead it represents a relation between the two sets he has identified.

The restricted conception of fraction emerged in questions in which the number of partition units is a multiple of the denominator of the fraction - see figure 1. Students established a correspondence between the denominator and the number of partition units into which the whole is divided that is, the cardinality of the set formed by the partition units. Consequently, the numerator is determined by the cardinality of the part. With this mental image, in situations in which these characteristics are not fulfilled, the pupil requires a redefinition of the given whole in order to adapt his own conception of fraction. He explains that the whole should be divided in as many equal pieces as indicated by the denominator and the numerator expresses the number of partition units taken into consideration. Some students do not develop a more abstract meaning of fraction, they retain this restricted conception and so face an obstacle to further understanding, in particular, for learning equivalent fractions.

The predominance of the cardinality of the part. In this category we grouped all those manifestations in which a dissociation of the components of the numeral or of the part-whole relationship was carried out and a tendency to assign a primary role to the denominator was revealed.

This source of failure was more often found in items corresponding to the discrete model and appeared in various forms.

In problems where it is necessary to carry out a partition of the whole, the denominator of the fraction assumes an important role in the strategy used to solve the question, whereas in items where a subdivided whole is provided the numerator is the focal point. Thus, the solution of these types of exercises requires a natural dissociation of the constituent elements of the numeral or of the part-whole relationship. These elements are connected again in order to rebuild the part-whole relationship and to express it in another language. During this process, some pupils focused their attention on the numerator or on the part omitting the other aspects which intervene in the given relationship, thereby inhibiting a necessary link between them. For this reason, the different forms that comprise this category were described in general terms as a centration phenomena. In figure 2, examples of the answers grouped in the category are given.

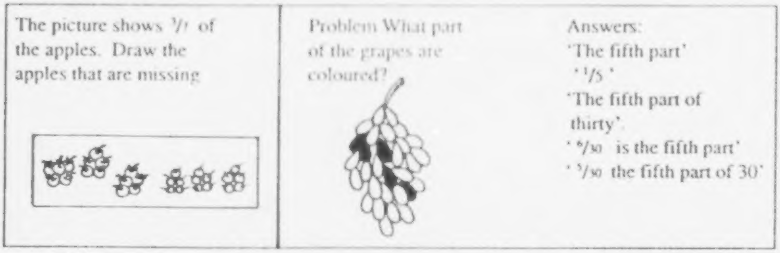

The picture shows $^{3}/_{7}$ of the apples. Draw the apples that are missing
Problem What part of the grapes are coloured?
Answers:
'The fifth part'
'$^{1}/_{5}$'
'The fifth part of thirty'.
'$^{6}/_{30}$ is the fifth part'
'$^{5}/_{30}$ the fifth part of 30'

Figure 2: Examples of answers classified in 'The predominance of the cardinality of the part'.

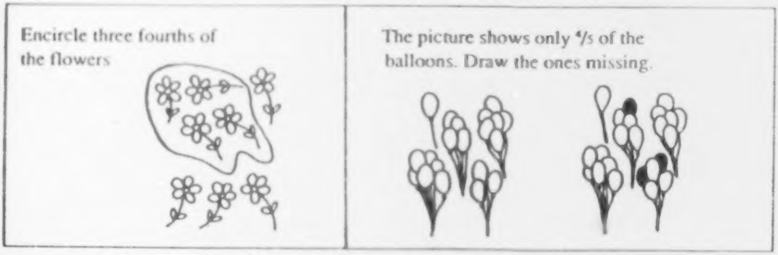
Considered of great significance was the following form of this category: the pupil assigns the number of partition units of the part to the denominator - see figure 2. These answers appeared in items that require a translation from a graphic language to a numerical one. This source of failure was identified in three different ways: expressions in ordinary language, arithmetic - symbolic representations and combinations of both languages. In all of these answers a unitary fraction in which the denominator corresponds to the cardinality of the part is associated with the part-whole relationship. It seems that in the connection between ordinary and symbolic language one could locate the origin of some of the difficulties.

The predominance of the denominator. This category contains all those answers in which a dissociation of the constituent elements of the numeral or of the part-whole relationship is done and a tendency to grant a relevant position to the denominator has arisen. Focusing his attention on the latter, the pupil deviates or omits the relation it had with the numerator.

The phenomena grouped in this category with its different manifestations, also emerge from a dissociation of the components of the numeral or of the part-whole relationship in items where this procedure is necessary for its solution.

Two main types of problems we identified in this category. One is associated with a perceptual centration induced by the features of the image and the other is related to specific meanings given by the students to the denominator.

The images that favoured a perceptual centration have the following characteristics: (a) configurations displayed in groups of objects, in which the number of groups coincide with the denominator, (b) pictures in which a subset of objects can be isolated from the rest of the configuration; the number of elements of the subset is equal to or bigger than the denominator. An example for such answers is shown in figure 3.


Encircle three fourths of the flowers

The picture shows only $^{4}/_{5}$ of the balloons. Draw the ones missing.

Figure 3.Examples of answers classified in 'The predominance of the denominator'

For the second type of problem the most significant error was to assign the denominator to the number of objects of the part in the graphed representation of the part-whole relationship.

The second and third source of failure which arose only in the solution of items linked with one of the two teaching models provided evidence of a differentiation of areas of cognition.

These were: (a) questions within the primitive Egyptian model and linked with partition of shapes, equality of amount of area and congruence of forms, essentially geometric problems, (b) for items linked with the discrete model, grouping and counting procedures are required, typically arithmetic problems.

Final comments

It seems likely that the pupil, subjected to a teaching process, constructs different conceptions of rational number, each of them associated with a teaching model. The notions he builds up are related to his own interpretation of the meanings embedded in the model and with the interrelationship he is able to establish with concrete referents. If the juxtaposition of the teaching models enables him to connect various conceptions, he articulates a conceptual net, building up in each stage a new mental image of the construct. The students' conceptual net is related to the one provided by the curriculum.

Teaching fractions with only one approach, the pie model, and taking for granted that students will be able to transfer their knowledge to other contexts and to establish relationships between other meanings of fractions, is a teaching strategy doomed to failure. Appropriate alternatives must be found to make the learning of fractions and rational number concepts more accessible to the pupils.

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Psychological difficulties in understanding the principle of mathematical induction

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Abstract

It has been found that high school students, after learning systematically the mathematical induction principle (and its application as a mathematical proof) are facing difficulties in understanding it genuine meaning. The main difficulty seems to be that the student is inclined to consider the absolute truth value of the inductive hypothesis in the realm of the relatively autonomous "induction step" $P(k) \to P(k)+1)$. Consequently, he cannot realize how a statement to be proven (the theorem) may become a premise in the structure of the proof itself.

Introduction

Let us remind briefly the principle of mathematical induction. This principle states the following: (a) If a proposition $P(n)$ is proven to be true for $n=1$ and (b) If one proves that for $n=k$ $P(k)$ implies $P(k+1)$, then $P(k)$ is true for every n. The variable ranges over the set of natural numbers. (In fact, the initial number on which the set is based may be different from 1 but commonly it is 1).

In the axiomatic form one gets:

$$[P(1) \,\&\, \forall k[P(k) \rightarrow P(k+1)]\ ] \rightarrow \forall n\, P(n)$$

Ernest (1984) describes schematically a typical induction proof as follows:

Theorem : P(n)
Proof : by mathematical induction
Basis : Proof of P(1)

Inductive hypothesis: assume P(k)
Induction step : proof of P(k+1) from inductive hypothesis.
(Ernest, 1984 p. 176)

The essential character of this kind of reasoning, as Poincaré has affirmed,is that "it condenses in a unique formula, an infinity of syllogisms" (Poincaré, 1906, p 20).

The use of the principle of induction as a proof, raises various technical difficulties to which we do not intend to refer in the present paper. But it has been found that even if the student is able to apply technically the method of mathematical induction, he very often, does not understand genuinely its meaning (Brumfiel, 1974;; Ernest, 1984).

Method

In order to get a better understanding of the difficulties the students encounter with respect to the mathematical induction principle, a research has been orgnized with high school students. One hundred and thirty eight students enrolled in four 11th grade classes with mathematics as a major topic, participated in the research. The students attended 15 lessons devoted to mathematical induction in the framewoork of their usual mathematics programme. After the conclusion of the series of lessons, the subjects were asked to answer to several categories of questions but, in the present paper, we will focus only on four questions referring to the inductive hypothesis.

The questions were:

1) Yaakov claims: "I have just proven a theorem using the mathematical induction method but, as a matter of fact I do not know for sure whether the theorem I have proven is really true because I relied on the induction hypothesis (the truth of the statement for a certain k) and I do not know if the statement is really true for this k".

Do you agree with Yaacov ?
Yes............. No..........
Justify your answer

2) Dani claims:

"With mathematical induction we prove a theorem represented by the basis and the induction step. But even if we have got the proof, the hypothesis is only an hypothesis and at the moment that the hypothesis is rejected the whole proof is no more valid".

Do you agaree with Dani ?
Yes........./No..........
Justify your answer

3) The "inductive hypothesis" is only an hypothesis. In your opinion is there any way to check whether the claim expressed by the inductive hypothesis is confirmed ?

Yes........../No..........
Justify your answer

4) Do you agree with the utterance: "In a proof based on mathematical induction, there is a flaw contained in the induction step. In this stage we assume, initially, that the statement is correct, in continuation we rely on that and afterwards we conclude that the statement is true" ?

Yes........./No.........
Justify your answer

All the questions were inspired by commentaries and problems raised by the students in their answers to preliminary questionnaires. We concluded that they express real misconceptions and that it is worthwhile to determine their structure and their frequencies.

Results

In the following lines we refer to the various justifications presented by the subjects in order to support their agreement (or disagreement) with the claims expressed in the questionnaire.

These reactions have been classified and we quote some examples of the categories obtained. The statistical data will be presented in a complete account of the research to be published later on.

Correct answers:

1) First type: schematical answers:

"After the two parts have been proven (confirmation of the basis and the proof of the induction step), the statement has been proven to be correct for every natural number greater than the basis. The hypothesis becomes a fact".

2) Second type: Explication of the chain process

"The inductive hypothesis is confirmed by the process of the mathematical induction. At the beginning, one confirms the claim concerning the basis. Afterwards, the truth of the induction step confirms the truth referring to the next natural number, and so on. This way one generates, strating from the basis, an infinity of correct statements. Consequently, the statement is correct for every natural number".

Incorrect reactions:

(1) The limited validity of the inductive hypothesis:

Examples

(a) "The statement has been proven if the inductive hypothesis has been proven.

(b) "The inductive hypothesis should be considered true till the contrary has been proven"

(c) "As long as the inductive hypothesis is true the statement is true".

(2) The truth of the inductive hypothesis is guaranteed:

Examples:

(a) "Through the proof (by mathematical induction) we do not prove the inductive hypothesis, because we know that it is true".

(b) "The inductive hypothesis is true and therefore we may rely on it".

(3) The truth of the inductive hypothesis cannot be proven:

Examples:

(a) "We suppose that the induction hypothesis is true but we cannot prove it".

(b) "It is totally impossible to prove the inductive hypothesis".

(4) There is no relationship between the truth of the inductive hypothesis and the truth of the steps of the mathematical induction:

(a) "It is possible to find that the inductive hypothesis is rejected and, despite this, the (inductive) proof is correct".

(b) "If the basis and the inductive step are true, this is enough in order to prove the statement, even if the inductive hypothesis is not correct".

(5) The truth of the basis confirms the inductive hypothesis

Examples:

(1) "If the truth of the basis has been confirmed one may assume that the number k, which appears in the inductive hypothesis, is the same initial number and then the statement is true for k".

(6) The truth of the induction step confirms the inductive hypothesis:

Example:

(1) "If the inductive hypothesis had not been true, we could not have proven the induction step".

We have found that only 28.3% of the students, who participated in the teaching program for mathematical induction, gave consistently correct answers. Among the others, 48.6% were inconsistent, 2.8% were consistent in their incorrect answers, and 20.3% made various types of other errors. The fact that about half of the students were inconsistent in their reactions (some correct, some incorrect) expresses the collision between the taught concepts and the intuitive biases.

Discussion

The psychological problem the student has to face is that the statement P(k), say $1+2+3....+k=\frac{k(k+1)}{2}$, appears twice in the inductive reasoning: as a statement to be proved and as a condition for the truth of the same statement.

What is difficult to understand is that P(k) (the inductive hypothesis) is postulated in the reasoning process (in the induction step) not as a proved fact but as an hypothesis - that is with its initial status. The difficulty is that the student has to build the entire segment of the induction step (if P(k) is true then P(k+1) is also true), on a statement which, itself, has not been proven and cannot be proven in this segment of the reasoning process.

As a matter of fact, we are absolutely not used to this way of reasoning. We may start from a given reality for formulating a certain theorem and then try to prove it by checking its consequences. Or we may start with a theoretical hypothesis and try to prove it by confronting it with a real situation. What we usually do not do is to check the implication itself, in itself, without any concern whatsoever for the truth of the two statements involved. It is like building a bridge in the air without any support on both its ends.

In fact, the mathematical induction connects, in its final stage, the two statements already proved, P(n) for n=1 and P(k) → P(k+1), and then starting from n=1 the infinite chain of syllogisms is proven. But the fact remains that the inductive step has to be proven, first independently as an autonomous statement.

As an effect of this situation the student is unable to consider the inductive hypothesis (P(k) in the framework of the induction step, as a statement for which the notion of absolute truth is not relevant. Consequently he adopts one of the following attitudes:

a) The truth of the inductive hypothesis is guaranteed.
b) The truth of the inductive hypothesis cannot be proven.
c) The inductive hypothesis has a limited validity (it is possible that, in certain circumstances, the inductive hypothesis does not hold)

In short, we suggest the following model for explaining the students' difficulties in understanding the principle of mathematical induction.

The induction step requires a proof on its own (as a temporarily autonomous implicative statement). The idea that one has to prove an implication $p \rightarrow q$ for which the problem of the objective truth of each of the two components, p and q, is totally irrelevant (in the realm of the induction step) seems to be intuitively unacceptable. This situation is complicated by the fact that the antecedent p includes the theorem to be proven. And then the student, being inclined to look for a truth value for the antecedent, is puzzled by the fact that the acceptance of the antecedent depends on the theorem which has to be proven.

According to the Piagetian theory, one of the main aspects of the formal operational stages is that the adolescent becomes able to manipulate logically, propositional structures in which implications play a fundamental role. But it seems that things are more complicated than predicted by the Piagetian theory. It seems that many adolelscents still tend to confer absolute truth values on each of the statements to be connected by implication, even when such truth values are irrelevant in the formal given conditions.

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Incipient "algebraic" thinking in pre-algebra students

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The goal of this study is to investigate the ability of pre-algebra students to make transitions from quantitative to qualitative arguments and viceversa, both in generalization and justification situations. We analyze the potential of some problem situations to provide an environment for such processes and the behavior of the individual student (or student pairs) in this situations. In the observations reported here, students were able, at least partially, to act at the qualitative level without symbolic algebraic language. However, there are situations in which the lack of symbolic tools hinders their performance

Introduction

The essence of mathematical activity consists of making generalizations (stating theorems) and justifying them (proving). Difficulties in generating and justifying generalizations from particular cases, may have different sources. There may be problems in "seeing" or establishing a general pattern, or in its justification. On the other hand, even when the student is able to visualize a pattern, and maybe even to give some kind of justification for it (be it verbal, visual, etc) he/she may fail in the symbolic manipulations required to establish and justify a general pattern in "mathematical language".

Mathematical language is a tool with a twofold purpose: it expresses the result of our thinking, making communication possible, and at the same time it fosters and stimulates thinking. Nevertheless, to learn to manipulate the language meaningfully seems to be a lengthy and difficult process. Consider the case of algebra. Several research studies show that students have difficulties in using algebra as a powerful means of capturing the mathematical "essence" of a wide variety of situations, and their handling of the language tends to be a technical (and frequently meaningless) manipulation of symbols according to some arbitrary rules (see, for example, Lee & Wheeler, 1986; Bell, 1988). In other words, many students, even when they manage to handle the algebraic techniques successfully, fail to see algebra as a tool for expressing and communicating generalizations and for justifying mathematical arguments.

One may be inclined to attribute student difficulties in understanding the meaning of algebra to the fact that most algebra courses, either directly or indirectly, tend to stress technical skills, neglecting the underlying semantics. However, instruction alone is only a part of the whole picture.

In the ongoing study, whose preliminary results we present here, we intend:

1- To investigate the potential generalization and justification processes of students who have not been exposed to formal instruction in symbolic algebraic manipulations. We do not want to make a case for the complete separation between algebraic syntax and semantics. We rather want to probe student thinking in situations in which they would otherwise have a strong temptation to be carried away by technical manipulations and algorithms.

2- To investigate student behavior in situations in which the lack of algebraic language is a barrier to progress.

We describe three of the problem situations we designed for our observations, and analyze them in the light of the above goals. Then we describe and analyze parts of the interviews with two pairs of seventh graders working on the problems, and discuss some preliminary findings.

The problem situations

Problem situation #1: Investigation of general patterns generated by subtracting two unit fractions with consecutive denominators.

Problem situation #2: Investigation of how the change in the sides of a rectangle (increase in length by one unit and decrease in width by one unit, or the increase of both length and width by a factor of three) affects its perimeter and its area.

Problem situation #3: Investigation of general patterns generated by the date numbers on a calendar sheet (say September 1988).

The problems were designed to be used according to the following framework.

Launch

The student is presented with specific examples, or specific examples are produced by him to understand the task.

Towards a "working generalization"

Producing additional examples	Producing or solving examples with large numbers	Solving "reversal" tasks

Towards an explicit generalization

Verbal description of the pattern observed	Symbolical description of the pattern observed

Towards a justification

We examine, for example, problem situation #1 in the light of the above framework. **Launch**: "Calculate 1/2 - 1/3. Calculate 1/3 - 1/4".

Towards a working generalization: By working generalization, we mean, an intuitive non-explicit sense of a general pattern, which can be elicited by certain tasks and applied in order to solve them efficiently. There is no need to verbalize the working generalization in order to use it. In the unit fractions problem, the following tasks were given to elicit a working generalization.

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Producing additional examples: Students are requested to direct their attention to certain common features of the task at hand and to proceed to generate additional numerical examples.

Solving examples with large numbers: "Calculate 1/212 - 1/213." Students are confronted with unfriendly numbers and are thus further pushed to use a more "general" strategy.

Solving reversal tasks: "Solve 1/? - 1/?= 1/110". The reversal task is another means of helping the student to focus on the general pattern.

Towards an explicit generalization: "Can you say anything about the pattern observed?". This reveals to what extent the students are able to verbalize what they have observed and also whether they can use algebraic symbolism, even though they have had no formal instruction therein.

Towards a justification: Students are requested to justify their observations and conclusions.

The subjects and the interviews

The subjects of the interview described in this paper were two pairs of seven graders attending a combined junior high and elementary school. Their mathematical ability was described by their teacher as average for one pair, and above average for the other.

The interviews were videotaped and lasted about one and a half hour each. During the interviews intense discussions took place between the two students in both pairs, whereas the interviewer's role consisted of supplying prompts and questions. The interviewed students were totally uninhibited, and acted naturally as if they were unaware of the camera.

Data and analysis

1. What is a "proof"?

This example is taken from the interview with K. and N., the two average seventh graders. Once a working generalization of problem situation #1 has been obtained, the issue of justifying the pattern was raised. First, 1/8 - 1/9 and 1/35 - 1/34 were presented as "proofs". K. even emphasized: "I think that any number is a proof, for example 1/5 -1/6."

The interviewer insisted that he wanted to be "absolutely sure" that the observed pattern was general. In response, K. gradually increased her "requirement" for a proof: "You need to check larger numbers as well.... you need at least five, six examples...". This was followed by a verbal statement as an explanation but still attached to a numerical case (1/8 -1/9): "Each time we have, let's say, 72 divided by 8 we get 9, and we divide by 9 we get 8.... one number less the other is always 1.... that's clear".

When asked whether numerical examples or a "general story" (justification of the pattern) are more convincing, the reactions of the two girls were rather mixed:

K: "I would tell a story. It's easier than showing many examples"

N: "I would tell a story, but also show several examples in order to prove"

I: "And if you could do only one of the two things?"

K: "I think that the story is more convincing, because I can explain why it happens"
N: "An explanation is more convincing, but for a proof you have to add [examples] - like to prove it to someone. He [a third person who has to be convinced] will say "up to here I understand [the general story] - now I want you to prove it to me." (Emphasized in the original.)
K: I think that to prove something means to show some examples and to explain, you have to explain why, like you must tell a story"

We can see that both K. and N. see the need for a general justification (called "the story" or "explanation") and at least K. can verbalize a semi-general explanation, but both of them are more convinced by numerical examples (called by them "proof" or "examples"). The meaning of the word "proof" for both K. and N. probably originates in everyday language where "proving" means "providing an evidence".

2. Visual versus numerical justification

Research shows that in mathematics there are personal preferences for visual or analytical thinking. In the case of the same pair of girls (K. and N.) one was clearly visual, whereas the other preferred numerical arguments. Therefore their interaction in problem situation #2 was very interesting. Problem situation #2 asked how the perimeter is affected by increasing one side of the rectangle by one unit and shortening the other by one unit.

N: (Simulating an attempt to break the two ends of her pencil) "If you take this line and break it and make a rectangle from it [after changing the lengths of the sides] you still have the same line. You cut from here [one end of the pencil] to there, and from here [the other end of the pencil] to there - and you still get the same."

K: "You calculate the perimeter and we always add this and that. And it makes no difference if you have 1 and 2 and then you have 3 and nothing. Because it's plus and it's not like division, where you cannot change numbers, change the order or something like that..."

We note that N, who strongly supported the case for the numerical examples in the previous problem, here seems to be able to "visualize" the general mechanism, independent of any numerical example. This phenomenon is described by Fischbein (1988) as follows: "visual representations contribute to the organization of information in synoptic representations and thus constitute an important factor of globalization. On the other hand, the concreteness of visual images is an essential factor for creating the feeling of self-evidence and immediacy. A visual image not only organizes the data at hand in meaningful structures but it is also an important factor guiding the analytical development of a solution."

When K. and N. were asked to compare the two ways of justification (visual and numerical), the following dialogue took place:

I: "Which explanation would you use to convince your classmates?"

K: "She [N] doesn't quite explain why. One could ask "how will you break [the pencil]?"

N: "Hers [explanation] is more theoretical, I would first show this [the pencil] and then give an example."

K: "You can also do it the other way. If they do understand mine [numerical method] then I would show them [the class] why. I think that your method is more general."

3. The lack of algebraic language as a barrier

This example comes from D. and O, the above average pair of students, working on problem situation #3: " Investigate general patterns in 2x2 squares in the following calendar sheet (September 1988). Almost immediately, D. and O. discovered many patterns, such as:

- The sum of the numbers in one diagonal (of a 2x2 square) is always equal to the sum of the numbers in the other diagonal.
- The difference between the sums of the the numbers in each row is 14.
- The difference between the products of the two numbers in each diagonal is always 7.

Next, the students were asked to justify their rules. The following are some excerpts from their justification process for their third rule, after their first attempt at testing the rule with numerical examples.

I: "You may try to work on this with letters, even though you have not learnt that yet...

O: "OK we have **a**, but we don't have any pattern here. One minute, there is a difference, I'll put **a**, **b**, **c**, **d** ... but what will happen with all these?" [Pointing to the numbers on the calendar sheet greater than the number they represented by **b** and smaller than the number they represented by **c**, and most probably referring to the fact that they are represented by consecutive letters although they are not consecutive].

I: "Can you reduce the number of letters...?"

O: "We can do **a** and **a+1** and **b** and **b+1**, 2 [for **a**] and 3 [for **b**] cannot be true, so we will put the same value". [To his friend] "Make a square, here **a**, here **a+1**..."

D: "Here **a+7**..."

O: "Here **a+7** because there is a difference of 7, and this one is 1 more, so it should be **a+8**"

D: "**a+7+1**"

I: "Can you explain what you have found? Why is the difference always 7?"

They write the products **a(a+7+1)** and **(a+1)(a+7)**.

O: [Expressing frustration] "It is an example, so we gave him an example, so what?"

I: "Will examples not convince a third person?"

O: "No! it is only an example!

D: "It's not true, we showed him numbers and then letters...

O: "No, we did not show him with letters why the difference is seven."

This pair of students understood the situation and could gradually produce an explicit generalization of the pattern using algebraic symbols. Whereas the generalization was clear to them, they were unable to produce a "decent" justification. D. was was quite happy with numerical examples. O. was aware that numerical examples alone are not enough, but he could not "show with letters" why the difference is 7. He reached the limits of his knowledge of algebra and at this point he needed the algebraic technique to produce the desired justification.

Concluding remarks

The above examples shed some light on thinking processes of seventh graders in problem situations. These processes show ability (and/or difficulties) of the individual and/or the student pair in the transition from quantitative to qualitative thinking and viceversa, both in generalization and justification situations.

On the whole, the students were able, at least partially, to act at the qualitative level without the symbolic algebraic language. However, there are situations in which the lack of symbolic tools hinders the performance at the higher level.

Like others (e.g. Bell, 1988) we think that such situations give meaning to the learning of algebraic symbols and techniques, because they develop student awareness of the need for algebraic tools.

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Development of the Process Conception of Function by Pre-Service Teachers in a Discrete Mathematics Course

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A group of 41 pre-service teachers began a course in Discrete Mathematics. The purpose of the course was to increase their understanding of mathematical concepts. At the beginning of the semester, observations suggested that most of the students did not have very much of a process or operational conception of function. The instructional treatment was based on a general theory of mathematical knowledge and how it is constructed by an individual. The particular methodology made use of computer activities designed to foster development of a process conception. At the end of the course there was evidence indicating that most of the students had developed a fairly strong process conception of function.

1 Introduction

Research in mathematics education has paid a great deal of attention to the concept of function. From Piaget [17] to the present time there have been many books, papers, and even at least one conference [18] devoted to this notion which is fundamental to all of mathematics.

A great deal of this work has been concerned with the study of student difficulties with particular aspects of functions such as the set of ordered pairs definition (for example, [16, 20]), a taxonomy of the function concept [6,8,9], the difference between operational and structural conceptions [22,23] and the historical development of the function concept [16,19,20]. Only a little of this work has been aimed at designing instructional treatments to overcome the profound difficulties that students seem to have in understanding functions (but see [1,5,25]), and even less has attempted to utilize a theoretical approach (but see [21]).

In this paper we present a preliminary version of an epistemology of the function concept and describe some of the results of using an instructional treatment based on this theory in one class. The main ingredient in implementing our theory is the use of computer activities to foster the constructions which the theory calls for.

2 Theoretical Considerations

2.1 General Theory

Our study of functions is based on a general theory derived from Piaget's concept of reflective abstraction [2]. We have used it in previous work to investigate mathematical induction [10,11] and quantification [12,13].

According to this theory, a person's mathematical knowledge is her or his *tendency to respond to a perceived problem situation by constructing or reconstructing mental schemas with which to deal with the situation.* In using the term *tendency* we are referring to the well-known phenomena [20] of students indicating in one situation that they seem to understand some mathematical idea, but being unable, in a situation often very close in time and content to the original, to display the same understanding. Our statement also reflects the notion that the response of a student to a problem is heavily dependent on her or his perception of the nature of that problem and this can often be quite different from the perception of the person who set the problem. The aspect of our statement that is most important for the present work is the construction and reconstruction of schemas.

A *schema* for us is a more or less coherent collection of (mental or physical) objects together with actions and internal processes which are applied to these objects. A schema is constructed by means of certain cognitive activities which include: *interiorization*, which is the construction of an internal process relative to a series of actions that can be performed or imagined to be performed on objects; *coordination*, which is the

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construction of a new internal process by combining two or more existing processes; *reversal*, which is the creation of a new process by inverting an existing process; and *encapsulation* which converts a process into an object that can be seen as a total entity and acted upon by actions or processes. A schema is reconstructed when a subject finds it inadequate to deal with a particular problem situation and so he or she projects it onto a higher plane of thinking by building new objects and processes. This construction and reconstruction (or projection) are the two aspects of reflective abstraction and a major thrust of our theoretical work is an attempt to extend this notion of Piaget to more advanced mathematical topics than he considered.

Our theory also relates to the work of others and we can mention A. Sfard who has studied operational vs. structural conceptions [22,23] as well as several authors who have considered the idea of epistemological obstacle [3,4,24].

2.2 Conceptions of functions

We consider four ways of thinking about functions. Over a number of years we have been making a very simple observation of students at various stages of their development. Without warning, we will ask a class of students to write down a statement of what they mean by "function" in a mathematical context and then to give examples of functions. Most of our taxonomy is based on these written responses, but on occasion, we have interviewed some of the students and asked them to try to explain their thinking in giving the response.

In a few cases, the response will not indicate any reasonable conception (for example, "a function is a social event", or "the reason or idea behind an operation" or "I don't know what a function is") and we call this a *pre-function* conception. For the most part however, it is possible to characterise the students' conception as one of, or a combination of, the following: *correspondence*, *dependence*, *transformation*, *total entity*.

This taxonomy is not very different from what has been obtained by other authors, for example T. Dreyfus and S. Vinner [9], but we find it helpful to explain it in terms of our general theory.

CORRESPONDENCE. There must be present a notion of variable, and two instances of variable are coordinated. Thus the subject, on one level of thought, might refer to "an equation relating two variables". There may be some suggestion that these variables are taking on values, but nothing is said about any uniqueness in the value of the second variable, given the value of the first. Thus the subject does not distinguish clearly between a function and a relation, and this might be reflected in the examples given. These examples are invariably restricted to algebraic (or trigonometric) expressions in two variables.

When projected onto a higher plane of thought, this conception becomes the familiar "set of ordered pairs with uniqueness to the right". A great deal of development must intervene between these two levels and that will involve some of the other conceptions that we are discussing. In particular, in order for the ordered pairs idea to be meaningful and not just a memorized statement, the subject must have constructed a process that "evaluates the function" by going from the first to the second component in an appropriate pair.

In either case, we consider that with this conception, the subject is thinking of function as an *object*. Thus, the process resulting from coordinating two instances of variable has been encapsulated.

DEPENDENCE. The subject refers to two variables and their values and takes cognizance of the value of one being determined by the value of the other. This can be indicated by use of terminology such as "independent variable" and "dependent variable", or the requirement that a process *yields* consistent values, but it is difficult to see this conception in the particular kinds of observation we are discussing. It was observed and analyzed at length by Piaget [17] and we have seen it in other situations.

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We consider dependence to be an example of a *process* conception of function, but we will not be concerned with this process in the remainder of the paper.

TRANSFORMATION. At a relatively low level of thinking, especially when examples of functions are restricted to algebraic expressions, the subject is able to substitute numbers for variables in the expression and perform calculations to obtain a numerical value. Often, not much more than this is understood when the examples are given in a form such as

$$f(x) = x^2 + 3$$

but this does tend to distinguish a transformation from a correspondence.

Projected onto a higher plane of thought, the subject is able to think in terms of taking a quantity (numerical or otherwise) and doing something to it in order to obtain a new quantity. Often this dynamic nature of the conception is explicit in the subject's words. The examples will go beyond algebraic expressions and even include algorithms for making the transformation.

When the transformation is restricted to substitution in an expression, we consider that the subject is thinking of a function as an *action*. Projecting this conception onto a higher plane consists of interiorizing the action to obtain a *process*. We see here what may be considered as an *epistemological obstacle* to the development of the function conception. Substitution in expressions is a useful and important activity at certain times. It can happen that the tendency to think of functions in those terms prevents the subject from advancing to a process conception.

TOTAL ENTITY. When the subject tends to identify the function with an algebraic expression, the idea of substitution can be weak or not even present. This can be suggested by the subject's words in giving and discussing an example such as

$$x^2 + 3$$

with neither $f(x) =$ or $y =$ in evidence. The subject is able to combine two or more such expressions and even relate the expression to a graph with some knowledge of the relationship between the form of the expression and the shape of the graph. What is missing at this level is any understanding of the relation between these two and the process that transforms a value of x into a new value.

In a sense, this version of the "total entity" conception of function is at the beginning of the development of the function conception. There is an object, but there is in evidence very little in the way of useful and interesting properties because these properties have not been constructed by the subject. Later in the development of the function conception, when the subject has reconstructed this conception onto a much higher plane of thought by constructing a process conception and a rich repertoire of properties and experiential phenomena, all of this retinue is encapsulated and function once more is a total entity, but now the subject is able to bring forth one or more aspects of a function to utilise in dealing with a problem situation.

In either case, we consider that with the total entity conception, the subject is thinking of function as an *object*.

3 Fostering the Development

We consider, with L. Steffe and E. von Glasersfeld [27], that the goal of teaching is to try to help students develop their conceptions into forms that are consistent with "adult conceptions" or, in our case, to help students construct function conceptions that embody the richness of phenomena, methodology, example, and application that mathematics has produced over the last few centuries. In terms of the above taxonomy, this means an object conception that encapsulates the notion of function as a process of dynamically transforming

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one quantity into a dependent quantity and embodies the idea of set of ordered pairs with uniqueness to the right as an expression of the process of going from the first component to the second component.

In this section we describe one aspect of what was done in a course for pre-service teachers to achieve this goal and observations of students' progress with respect to the development of a process conception of function in the sense of transformation of quantities.

3.1 The class

The class consisted initially of 41 students of whom 30 were pre-service high school mathematics teachers. The remainder were preparing to be elementary and/or middle school teachers. The students were all in their second or third year except for one senior and three graduate students. All but one of the students stayed for the entire semester. Class attendance was generally high.

Most of the students had taken one or two years of college mathematics (generally calculus) so that their formal background in mathematics was fairly standard for third year education students. Their understanding of mathematics however was, by their own estimation, the general opinion of mathematics faculty members, and our informal observations as the semester began, not very sophisticated. Many expressed strong anxiety feelings with respect to mathematics. The situation was even worse with respect to the use of computers and, with some exceptions, those who had used computers seemed even more nervous about it than those for whom it would be a new experience.

3.2 The method

The class met twice per week for 75 minutes each. One meeting was in a computer lab and the other was in a standard classroom.

The overall approach was to have the students perform tasks on the computer that were designed to foster the specific constructions with respect to functions (coordination, interiorization, encapsulation) described in the preceeding section. Class sessions consisted of general discussion, paper and pencil tasks, and explanations from the instructor. The design of class activities was based on assumptions about what cognitive development was taking place as a result of the experiences that the students were having with computer activities.

The topics for the course were: number systems, propositional calculus, sets, functions, and mathematical induction.

3.3 The course and observations of students

For the purpose of describing the treatment of functions in the course and our observations, we can consider four periods.

FIRST WEEK. In the very beginning of the course we tried to find out something about the students' initial conceptions of functions by asking them in class to write down what they mean by a function and to provide an example. We call this **Observation A.**

WEEKS 2-8. In the first part of the course the students learned how to operate the computer system and became familiar with the programming language **ISETL** that was used throughout the course. They studied representations of integers, operations on integers (mod, gcd, etc.), propositional calculus, and sets. There was no explicit study of functions during this period, but an effort was made to point out where specific mathematical constructs they worked with could be interpreted as functions and, as much as possible, the language of functions was used where appropriate.

At the end of this period we made two observations of the students. **Observation B** was essentially a repetition of **Observation A. Observation C** consisted of giving the students a list of 23 situations and

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asking them to describe each situation in terms of a function when that was possible. They were given a few days to work on this at home.

WEEKS 9-13. This five week period was the main study of functions in the course. Aspects of functions that were studied included representing a function in ISETL as a func (that is, a procedure for computing it), an smap (that is, a set of ordered pairs with the uniqueness condition) and as a tuple (that is, a sequence); representing functions mathematically; writing programs to convert from one representation to another; properties of, and operations with, functions that emphasize the process such as evaluation, inverse, one to one, onto, composition and other binary operations; and applications of functions such as data bases.

Our feeling in designing this instructional treatment was that the act of writing an ISETL func to implement a function and thinking about what this procedure is doing as operations such as evaluation are performed would tend to get students to reflect on the actions involved and to interiorize those actions into a mental process. In this way, for example, the action of substitution could be reconstructed by the students into the process of function evaluation. Constructing a func to represent the composition of two function by simply having one procedure call another, writing procedures that converted a function (on a finite set) to its inverse, constructing computer tests for equality of functions were all expected to contribute to and reinforce an emerging process conception of function.

At the end of this period, we made Observation D in which the students were given a set of written questions in a situation that was test-like except that they were not told anything in advance that would suggest that they should prepare for it, nor was it to have any effect on their grade. In this way we tried to get a little closer to their "permanent knowledge".

There were three questions on this "test". The first asked them to use a function to organize the data concerning the occurence of vowels in the sentence

The quick fox jumped over Farmer Brown's lazy dog

They were also asked to represent this function in ISETL, to convert that representation to a different ISETL representation, and to explain whether some other representations were reasonable. The second question presented them with an ISETL func that called another func and returned a function represented as an smap. They were asked a number of questions about evaluation of this complex composition. The third question was suggested by Ruhama Even (14) and asked them about the importance of the uniqueness condition in the set of ordered pairs definition of function.

Observation E consisted of interviewing a selection of 7 students about their responses to Observation C which was made before the study of functions began. They were shown their responses and given an opportunity to change them or "defend" them.

LAST TWO WEEKS. The last portion of the course was about mathematical induction. Although this did not concern functions per se, the role of proposition valued functions of the positive integers was emphasized.

At the end of the course, Observation F was given as an examination in which there were several questions about functions. Some were fairly difficult such as to prove or disprove that the pointwise sum of two one to one functions is again one to one and similarly for onto. Other questions concerned the function process in composition, a function whose domain and range are sets of functions, and using the function process along with composition to express information extracted from a data base.

4 Results

There is too much data to present in the space available here. A longer work that is in preparation will organize the information and present it as quantitatively as possible. Here we restrict ourselves to some comments and tentative conclusions specifically about the development of the process conception that can

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be drawn from some of the student responses to our observations.

Observations A and B. There were two main differences between these observations and they could represent an effect of the 8 weeks of computer activities in which the concept of function was not emphasized, but was definitely present.

The first difference was that in Observation A, many of the students showed a very weak conception of function and 13 of 30 responses suggested the pre-function stage. Almost all of the responses (18 out of 20) that indicated a transformation conception referred to a substitution of values for variables and did not suggest any thinking about process. In Observation B, however, 9 of the 13 students who gave pre-function responses now gave a response that indicated some function conception, most in the form of a transformation. The latter responses increased to 28, but still, all but one was about substitution.

The second difference had to do with the richness of examples. In Observation A, 32 out of 39 examples had no expression other than an algebraic or trigonometric calculation whereas in Observation B, more than half or 57 out of 108 were about other situations such as a boolean valued function of the integers, a computer procedure, and so on.

Observation C. In this observation which was taken just after Observation B, a process conception seemed to be emerging in the minds of most of the students. Consider for example, the question of using a function to describe the following situation.

$$[2^n + n^5 : n = 1, 2, \ldots, 100]$$

which represents a finite sequence of numbers. Out of 30 students, 22 referred explicitly to some process that took an arbitrary positive integer from 1 to 100, performed a calculation and returned a result. An additional 5 students gave responses that could possibly be so interpreted.

This observation also produced one of the more interesting and surprising results by way of a spontaneous conception of function. There were 3 (out of 23) situations (to be described in terms of functions) which represented an equation in two variables x and y. In two of them it was possible to solve for one variable in terms of the other. This, or an invocation of the implicit function theorem was our intention. Instead, 13 students said that this was a function of two real variables that returned the value true or false, depending on whether the pair satisfied the equation. Of these, 7 gave such a response on all three and another 2 gave it in two of the three situations. It is important to note that nothing like such a function was ever mentioned in this course, nor can we find any place in the students' other work in which they came across it.

On the other hand, there were a fair number of indications of difficulties. A prevalent one may have been an epistemological obstacle resulting from the study of sets. The students learned about expressing sets in a form like

$$\{\text{expression in } x \ : \ x \in S \mid \text{condition}\}$$

and when this was used, many tended to think that this set up a function with x as the independent variable and the expression as the value. They seemed unaware of the several mathematical difficulties in such an interpretation, not the least of which is the absence of a process.

Other difficulties had to do with identifying a function with its set of values, unwillingness to use y as independent, and x as dependent variable, and seeing a process in a geometric or physical situation.

Observation D. About 2 weeks into the 5 week study of functions, a definite process conception appeared to develop for most students. In this test, 26 out of 37 students used an ISETL construct to obtain a process to compute or record, for each vowel, the number of times it occured in the sentence. 19 of them were able to find a second construct that did the same job and about half of them seemed to understand why other constructs would not be adequate.

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One of the questions on this observation asked them to think about evaluation of a function defined by coordinating several processes in a very complicated way. The result was a function F for which $F(c)$, c a positive integer, was a function defined as a set of ordered pairs; $F(c)(b)$, b one of $1,2,\ldots,10$ was a character string; and $F(c)(b)(a)$ was the a^{th} character of the string. The students were asked to evaluate $F(3)(7)(4)$; find a such that $F(4)(3)(a)$ is the character "r"; find b such that $F(2)(b)(1)$ is the character "e"; and find c such that $F(c)(1)(2)$ is the character "a". Again, they had never done problems of this kind in class and we considered that having a strong process conception of function would help them figure out on their own how to solve it. About 20 of the 37 students solved it correctly.

On the question asking about the uniqueness condition in the set of ordered pairs definition of a function, 27 out of 37 tried to relate it to the requirement that there be a specific process and 21 of those seemed to succeed.

Observation E. After the study of functions was completed, the seven students who were interviewed generally confirmed that they were using a process conception to decide whether a function was present in the situations of Observation D. Often, without prompting, they changed the response they had given earlier and seemed to see where they had been in error. These interviews will be analyzed more fully in a subsequent paper.

Observation F. The final examination included four questions in which the students had to work with the process of a function obtained by combining the processes of two given functions by composition, pointwise addition or pointwise multiplication. It also included a problem on which they had to interpret certain questions about a data base in terms of a function process. None of the questions were exactly the same as problems they had worked on during the course and some were quite different. We considered the exam to be rather difficult for these students.

On the first group of questions, the overall score of 40 students was about 62%. The score on the data base question was about 74%.

5 Conclusions

It seems reasonable to conclude that by the end of the course, most of the students had developed a process conception of function. This conception appeared to be strong enough to use in new situations, was available for applications such as data bases, and even remained relatively stable when students may have been disconcerted by being faced with a difficult problem. The early observations suggest fairly strongly that such a conception was either not present or fairly weak at the beginning of the course. Thus we may suggest that the students process conception did develop during this semester.

We cannot, however, be sure of why this happened. It could have been the result of a good rapport between students and professor, other experiences that the students may have been having at the time, or merely that there is a natural development of the process conception of function in an individual and this happened to be occuring at this time.

All we can suggest is that our results are not inconsistent with the possibility that using computer activities based on our general theory is an effective means of fostering development of the process conception of function. This in itself encourages us to continue with our approach in an extensive study of how students may come to understand functions and what might be done in a classroom to help.

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Proceedings

Volume 1.

p.2
BEHARIE M, NAIDOO Y. add : "in a multicultural school"

p.3
add :
DUBINSKI E., HAWKS J., NICHOLS D. Development of the process conception of function by pre-service teachers in a discrete mathematics course..................................291

Volume 3.

p.288
add :

HAWKS Julie Math Dept. Purdue University WEST-LAFAYETTE IN 47907 USA	NICHOLS Devilyna Math Dept. Purdue University WEST-LAFAYETTE IN 47907 USA

Programme

p.6
BEHARIE M., NAIDOO Y. : add "in a multicultural school".

p.14
add :
BECHARA SANCHEZ L. : Transformations géométriques - homothétie et similitude

p. 15
add:
LOPES A.J. : Enseignement à distance : un projet de formation et de recyclage de professeurs.

MASHEGO T.A.: Mathematics teaching in black south african schools - a psychological perspective.

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